Dispersive shocks in 2+1 dimensional systems

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Outline

- Introduction
- Shocks in dKP solutions
- Dispersive shocks in KP solutions
- blow-up in gKP
- Semiclassical DS II
- Dispersive shocks and blow-up in DS II
- Outlook

Korteweg-de Vries equation

$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0 \qquad u_0 = -\mathrm{sech}^2 x$$

$\epsilon = 0.01$

Korteweg-de Vries equation

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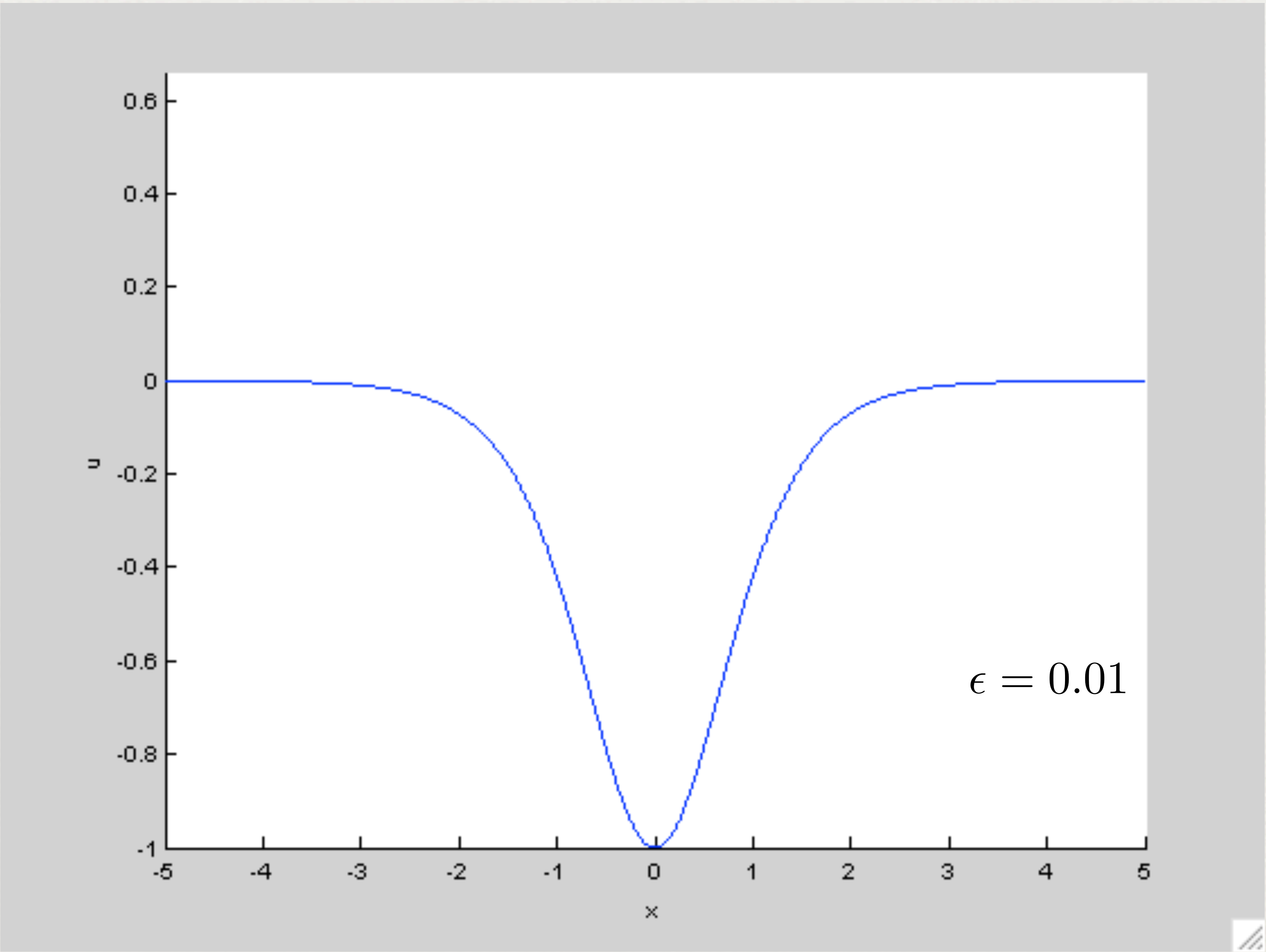
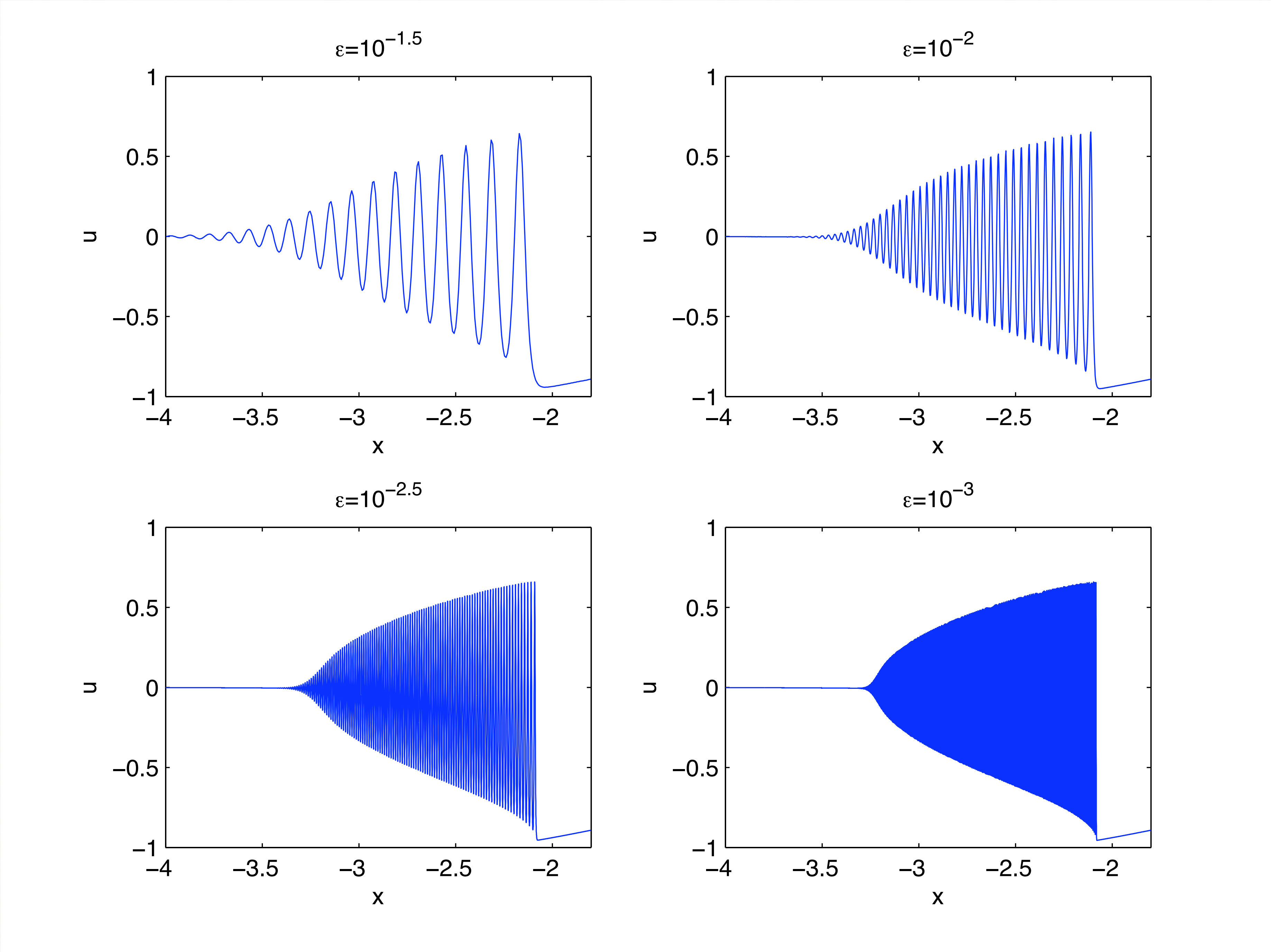
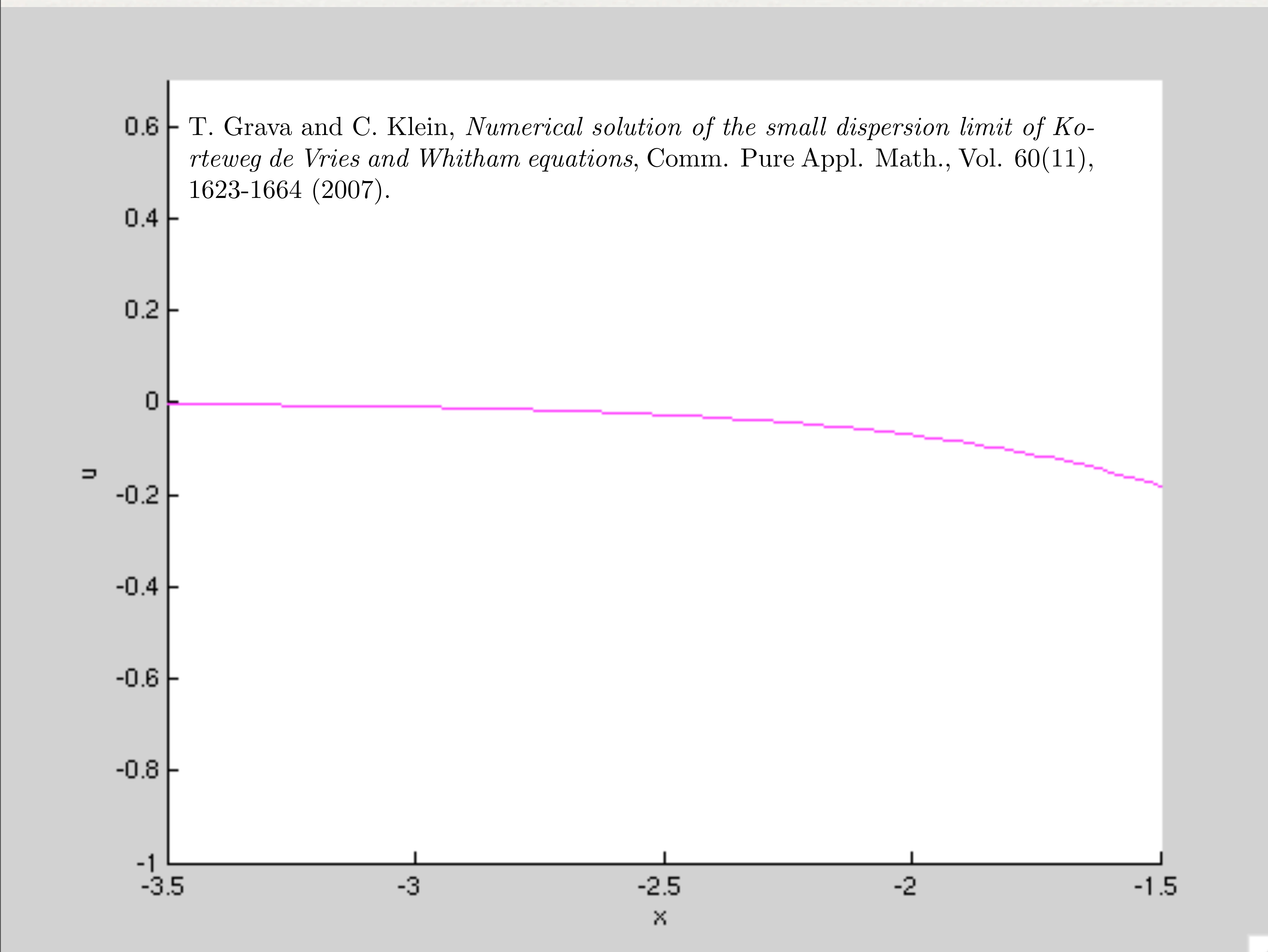


Different values of ε



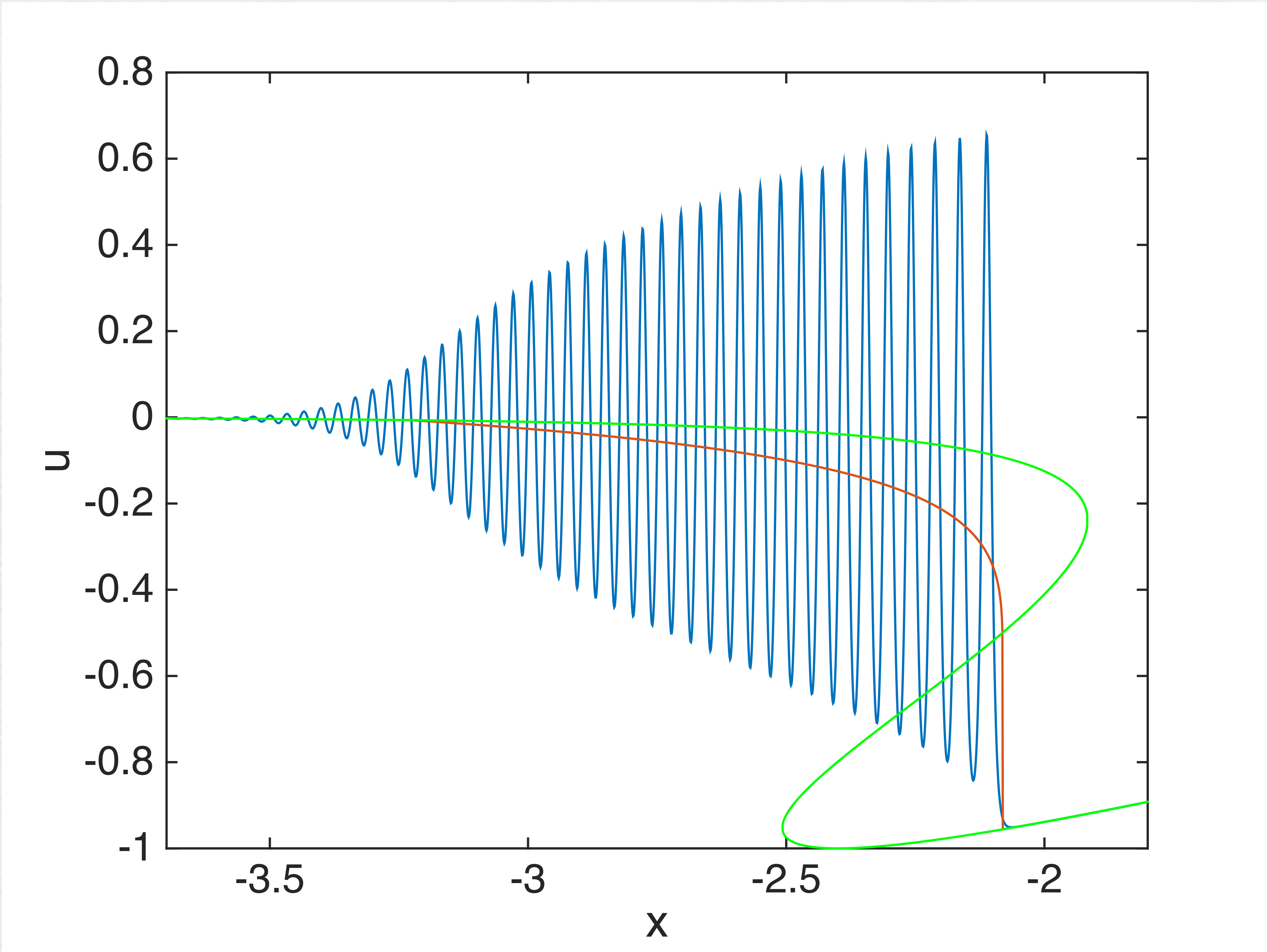
$$t = 0.4$$

KdV and asymptotic solution



$\epsilon = 0.01$

KdV, Whitham averaging and weak Hopf solution



Point of gradient catastrophe

- Dubrovin: universal behavior, solution near breakup characterized by cubic, $u \sim x^{1/3}$, multi-scale expansion
- solution given by fourth order equation, generalization of Painlevé I (x_c, t_c, u_c: quantities at breakup)

$$-6TF + F^3 + FF'' + \frac{1}{2}(F')^2 + \frac{1}{10}F'''' = X$$

$$u(x,t,\epsilon) = u_c + \left(\frac{\epsilon}{k}\right)^{2/7} F(X)$$

where

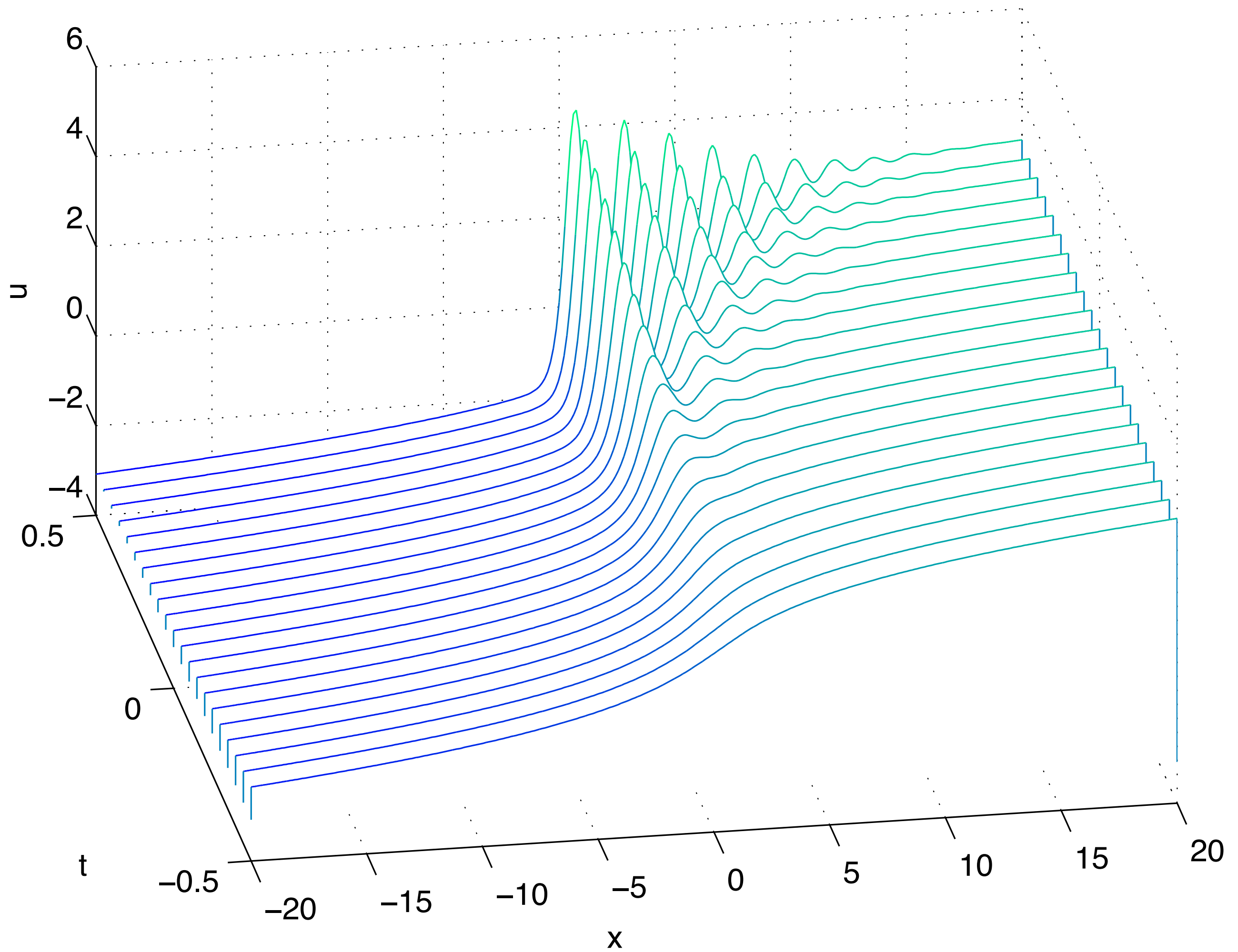
$$X = -\frac{1}{\epsilon}\left(\frac{\epsilon}{k}\right)^{1/7} (x - x_c - 6u_c(t - t_c))$$

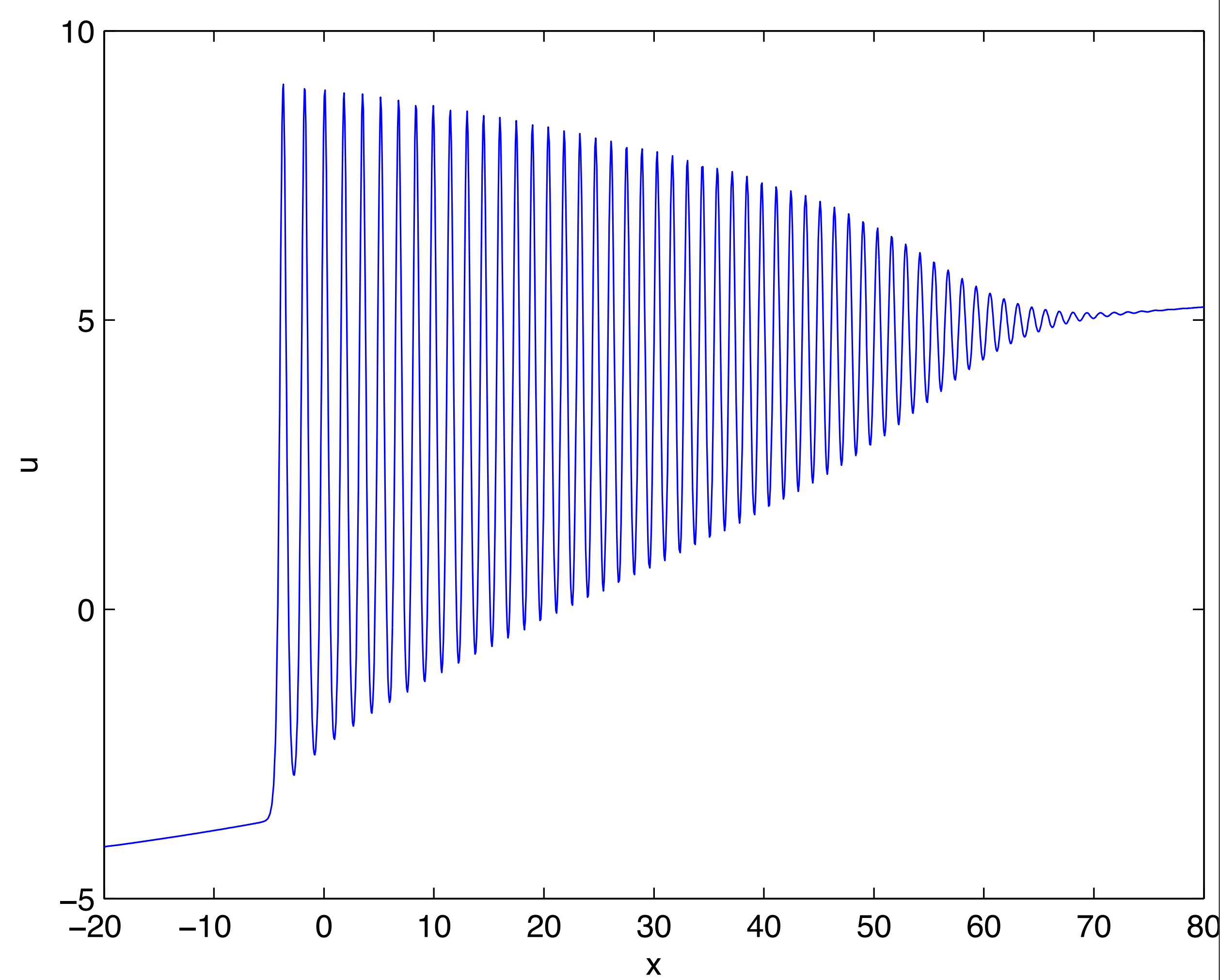
$$T = \frac{1}{\epsilon}\left(\frac{\epsilon}{k}\right)^{3/7} (t - t_c)$$

PI2 solution on the real axis, time dependence

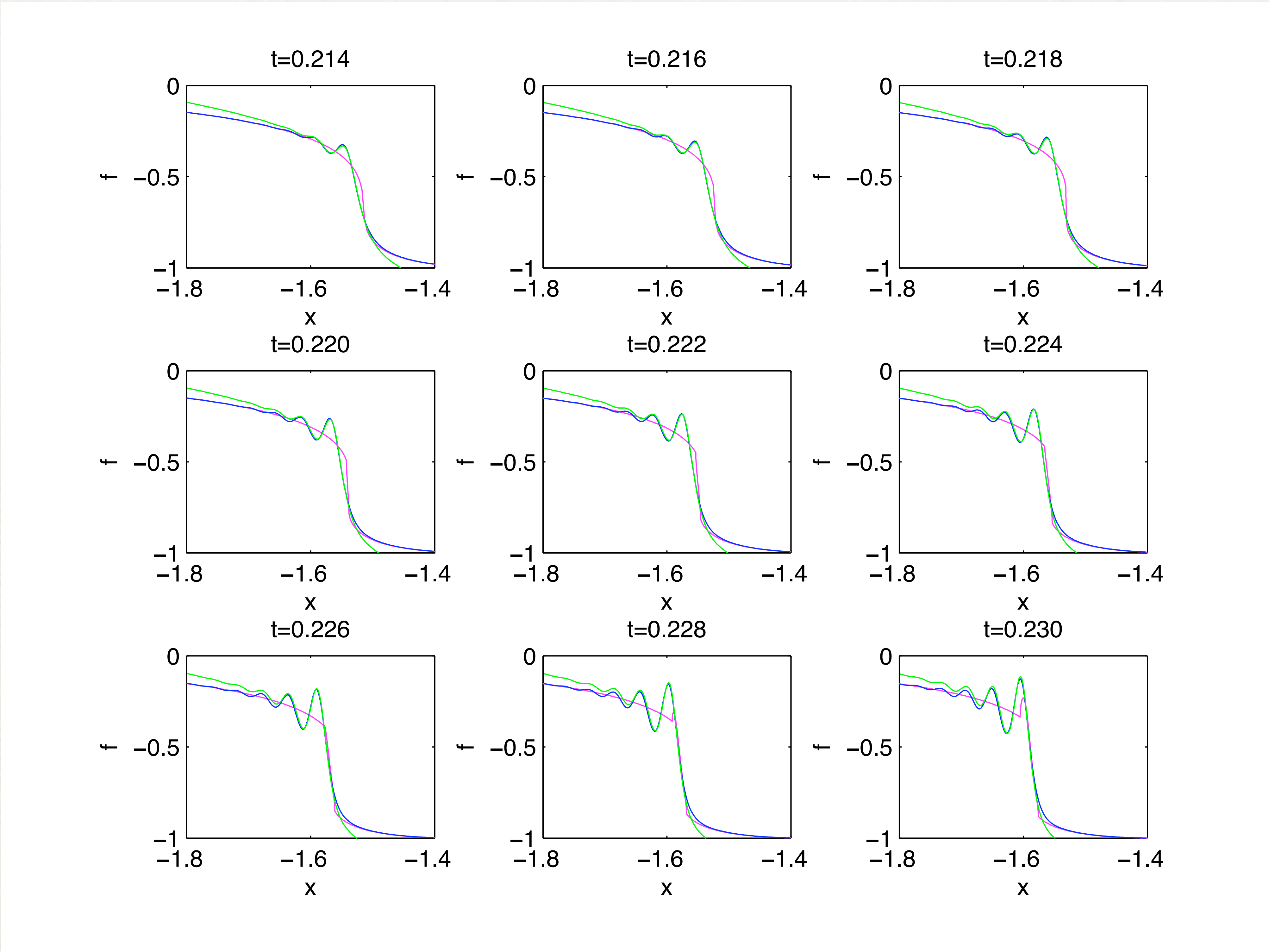
T. Grava, A. Kapaev, and C. Klein *On the tritronquée solutions of P_I^2*, Constr. Approx. 41, 425–466 (2015).

$t = 2$





KdV and PI2 near breakup

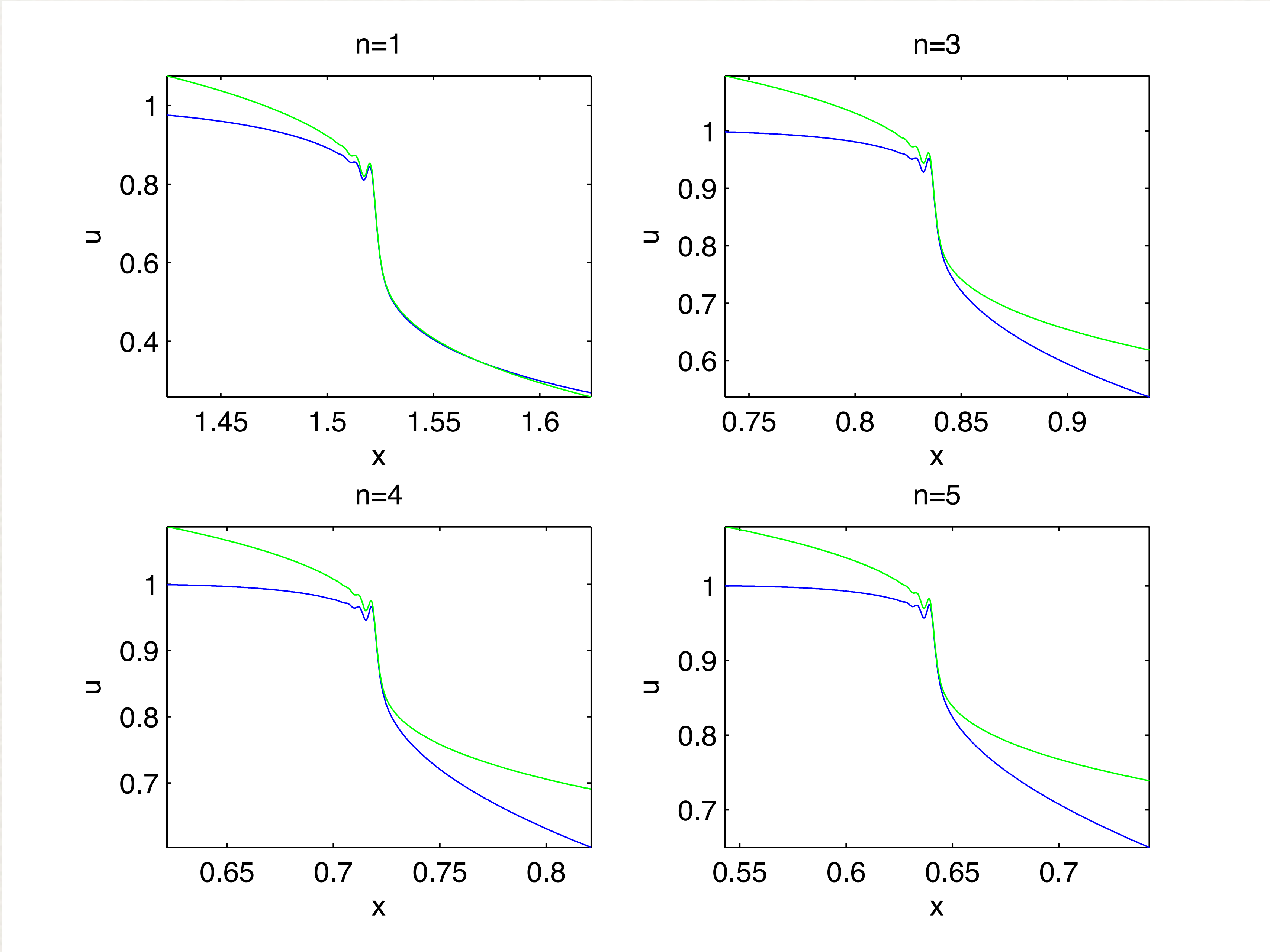


Non-integrable PDEs

generalized KdV equation

B. Dubrovin, T. Grava and C. Klein, *Numerical Study of breakup in generalized Korteweg-de Vries and Kawahara equations*, SIAM J. Appl. Math., Vol 71, 983-1008 (2011).

$$u_t + u^n u_x + \epsilon^2 u_{xxx} = 0$$



$$u_0 = \mathrm{sech}^2 x$$

gKdV, small dispersion

$$u_0 = \mathrm{sech}^2 x, \quad \epsilon = 0.1 \qquad n = 4$$

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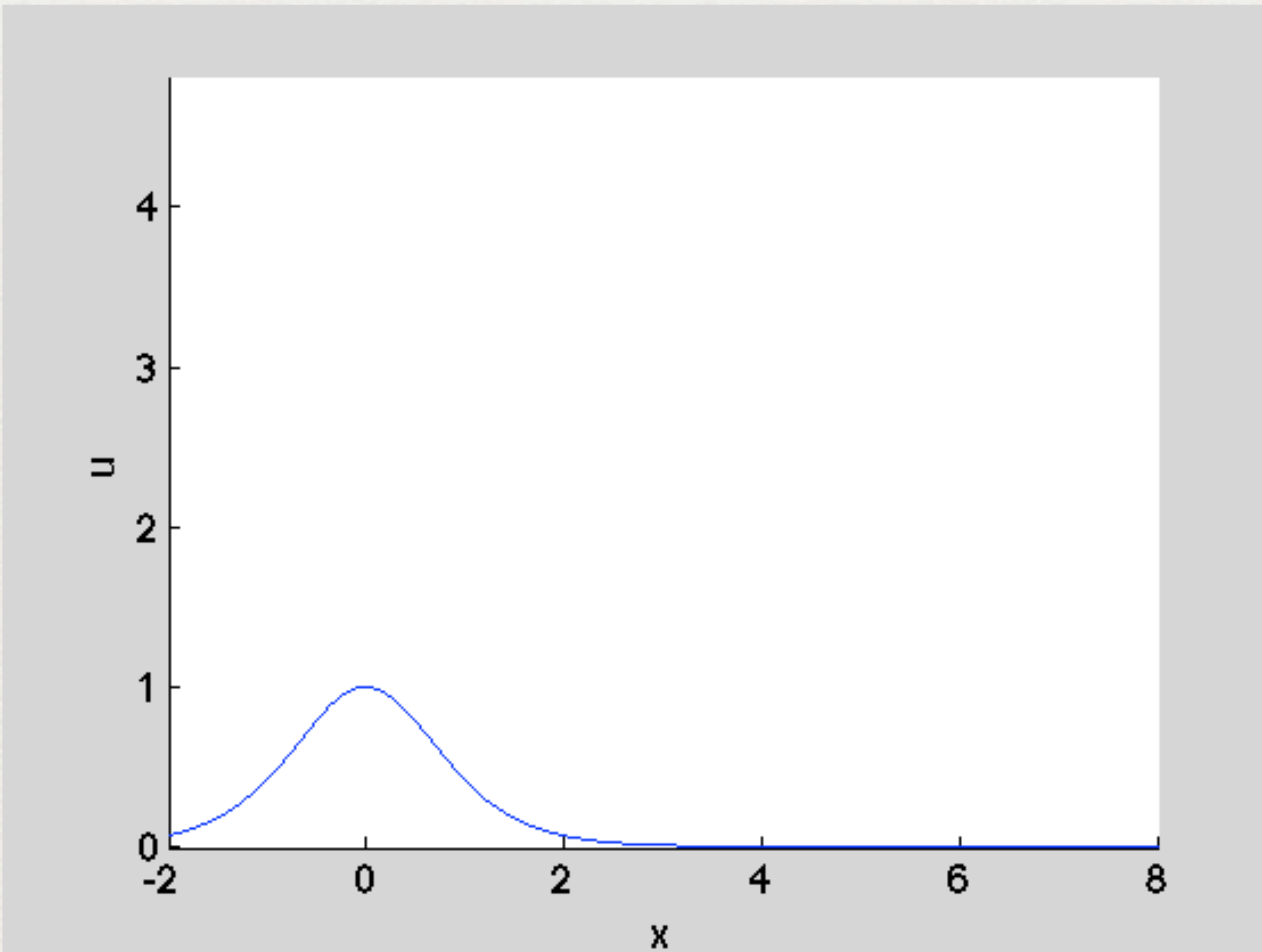

gKdV, small dispersion

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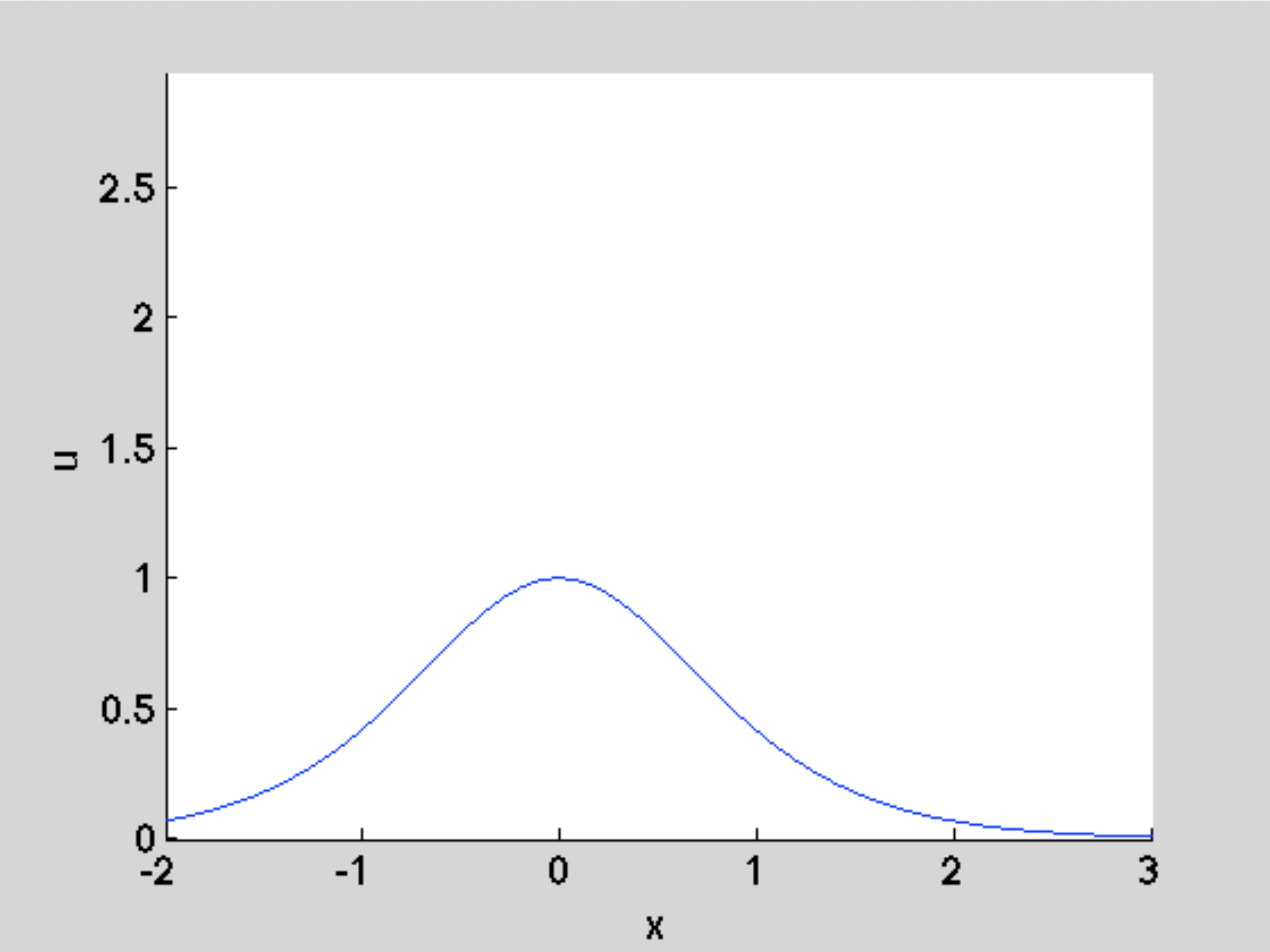
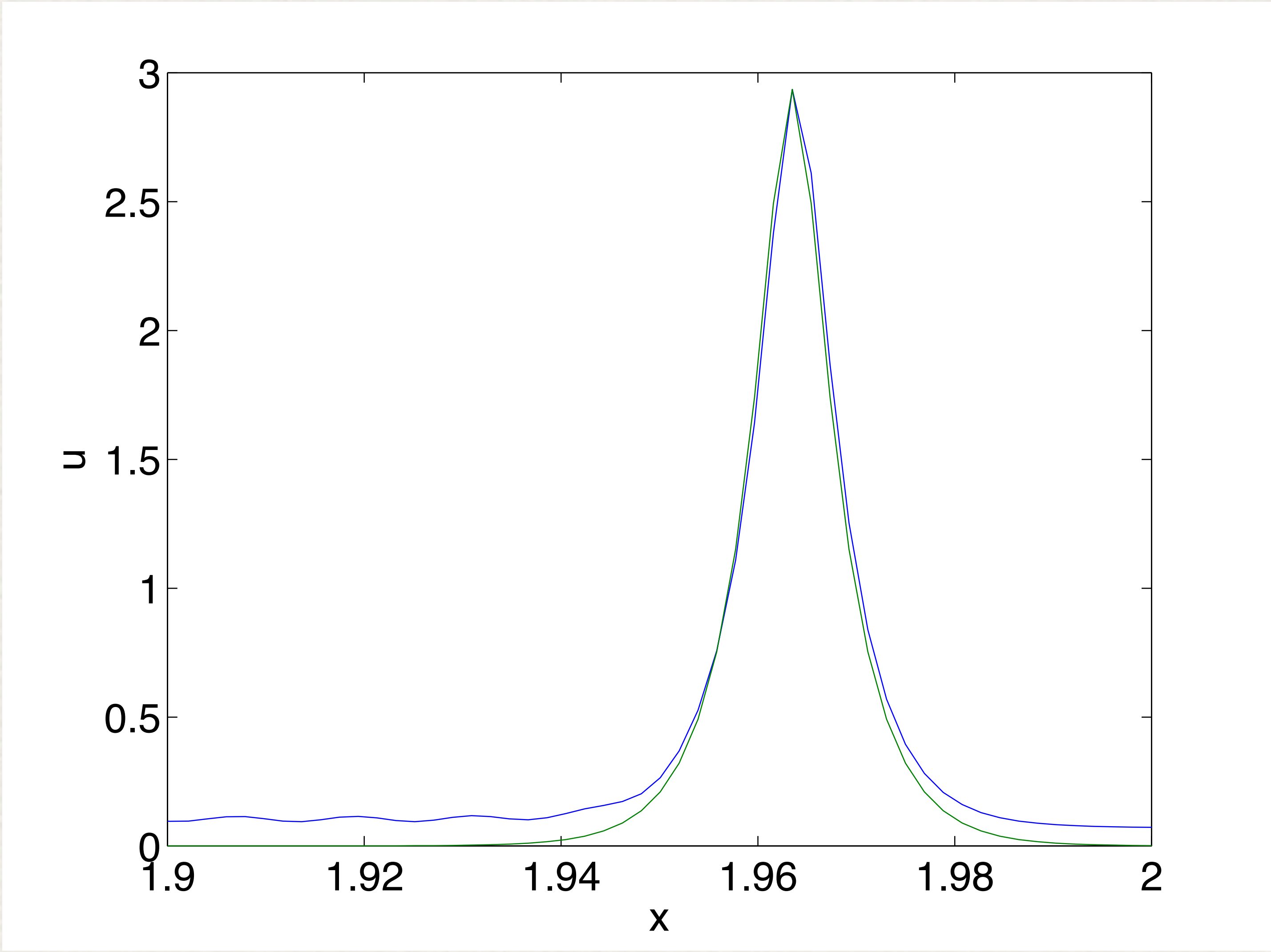


Fitting to rescaled soliton

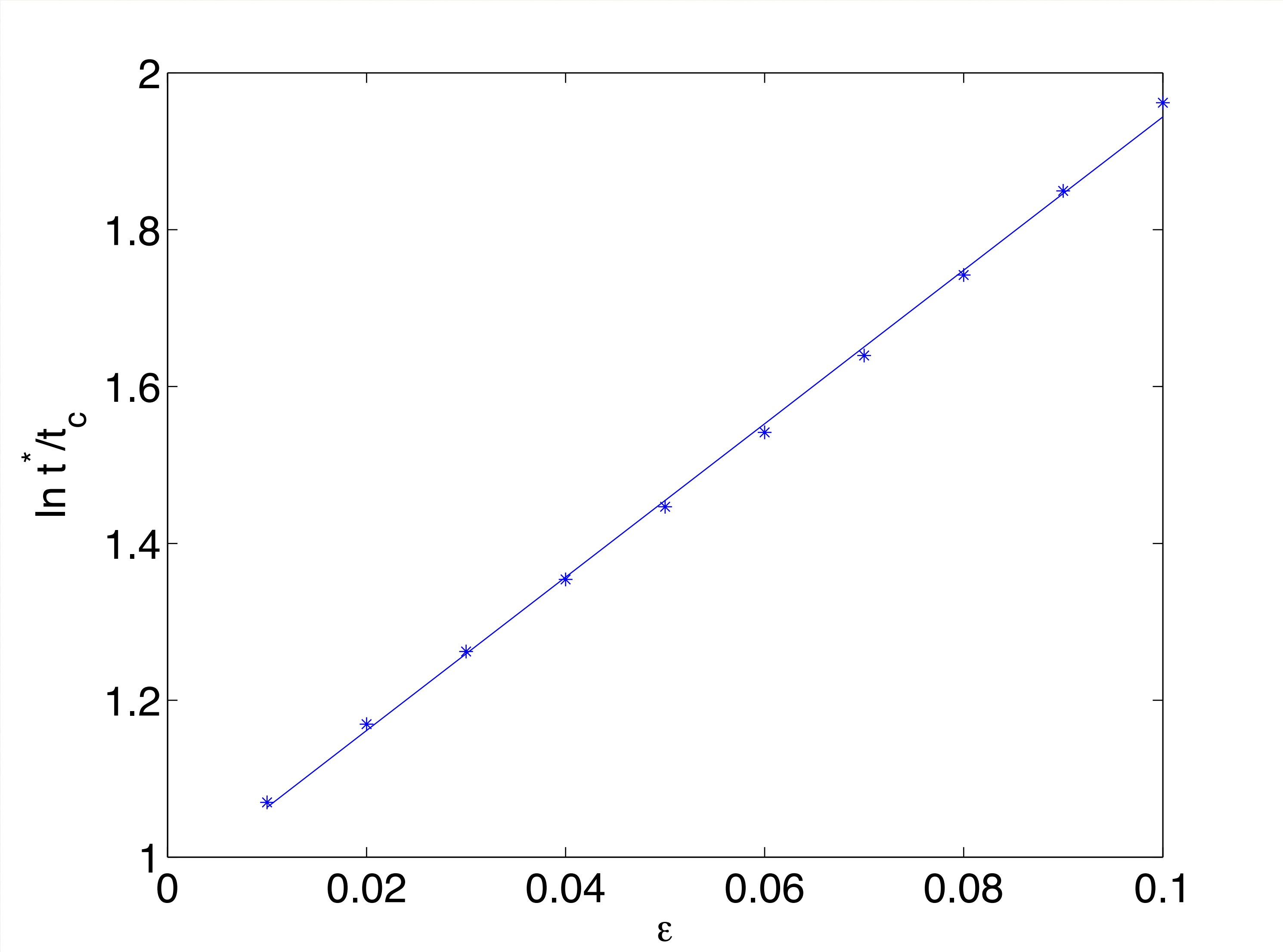
- Martel, Merle, Raphaël 2012: selfsimilar blow-up, blow-up profile dynamically rescaled soliton

C. Klein and R. Peter, *Numerical study of blow-up in solutions to generalized Korteweg-de Vries equations*, Physica D 304-305 (2015), 52-78



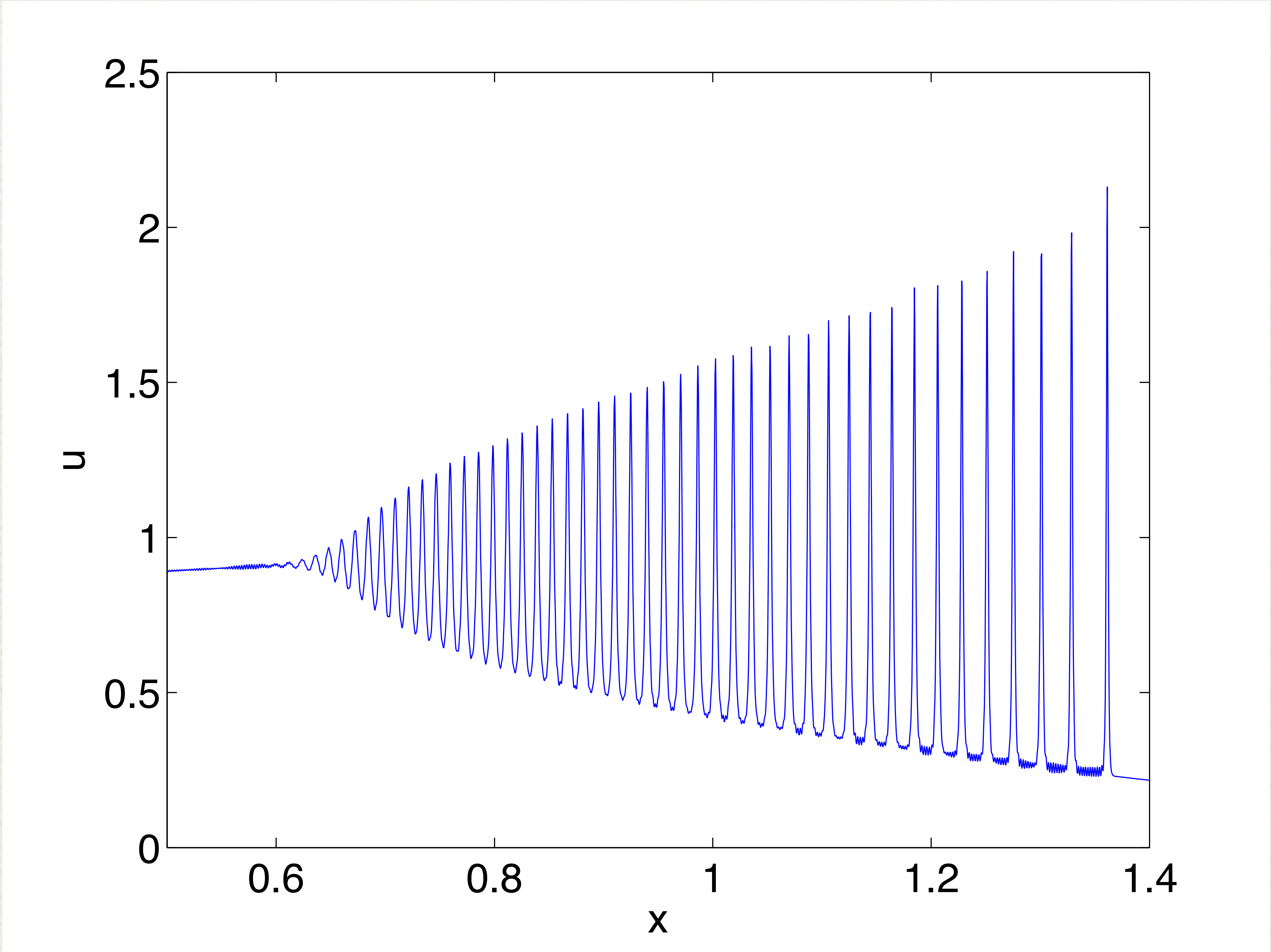
Scaling

- blow-up time t^* always greater than critical time t_c of Hopf ($\epsilon = 0$)
- exponential dependence of blow-up time t^* on ϵ, finite number of solitons appear before blow-up, fastest blows up
- universality?



gKdV, small dispersion

$$u_0 = \mathrm{sech}^2 x, \quad \epsilon = 0.001 \qquad n = 4$$



Kadomtsev-Petviashvili equation (KP)

- nonlinear dispersive waves on the surface of fluids, essentially one-dimensional propagation of the waves with weak transverse effects
  KP I ($\lambda = -1$): strong surface tension,
  KP II ($\lambda = +1$): weak surface tension

$$\partial_x \big(\partial_t u + u \, \partial_x u + \epsilon^2 \partial_{xxx} u \big) + \lambda \, \partial_{yy} u = 0, \quad \lambda = \pm 1,$$

- evolutionary form

$$\partial_t u + u \, \partial_x u + \epsilon^2 \partial_{xxx} u + \lambda \, \partial_x^{-1} \partial_{yy} u = 0, \quad u\big|_{t=0} = u_{\mathrm{I}}(x, y).$$

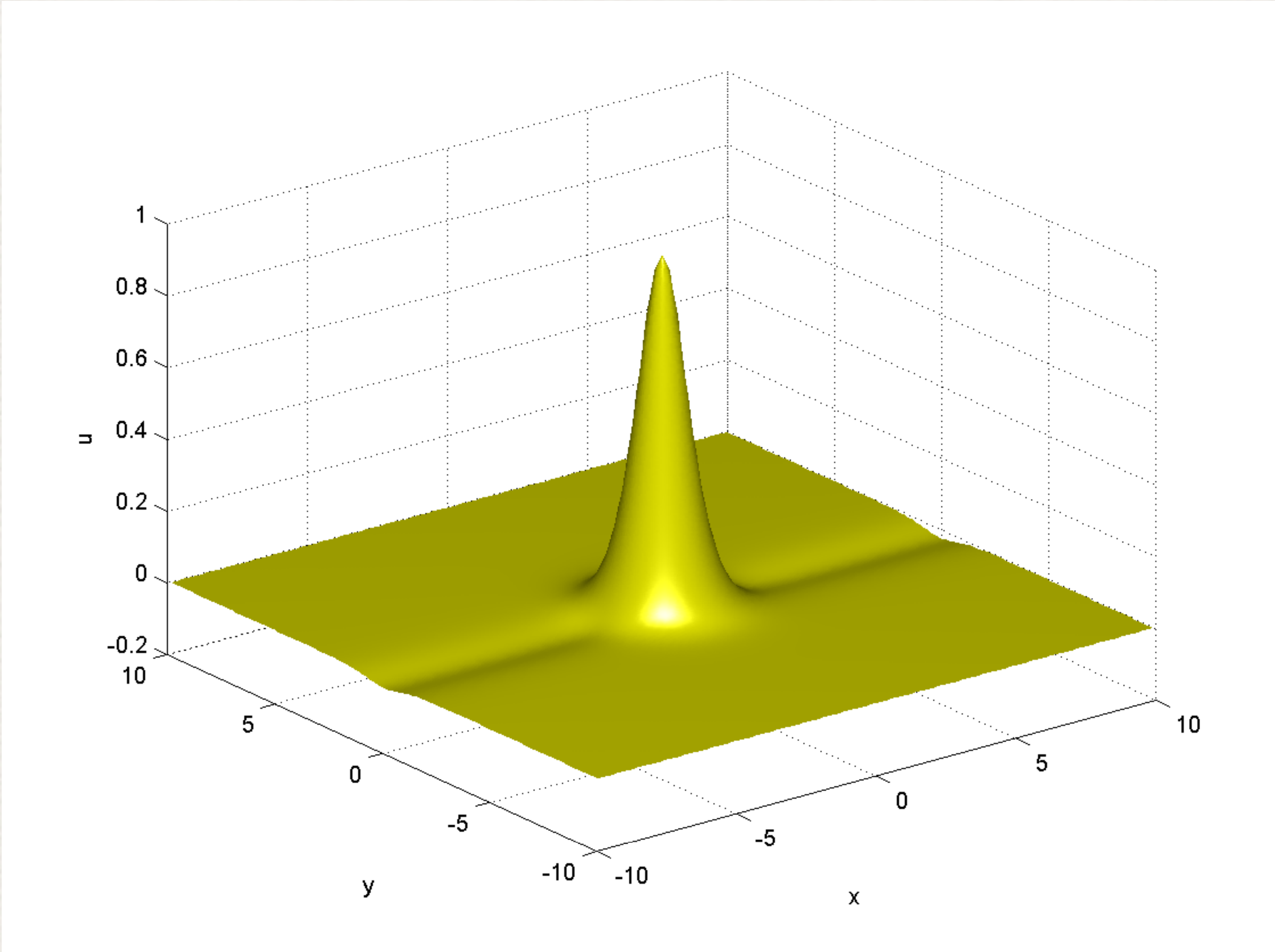
anti-derivative (Fourier multiplier)

$$\partial_x^{-1} f(x) := \frac{1}{2} \left(\int_{-\infty}^{x} f(\zeta) \, \mathrm{d}\zeta - \int_{x}^{+\infty} f(\zeta) \, \mathrm{d}\zeta \right),$$

Consequences of nonlocality

- constraint on initial data $u_0(x,y)$: $\int_{\mathbb{R}} u_{0,yy}(x,y)dx = 0$,
  - satisfied: solution to Cauchy problem smooth in time,
  - not satisfied: constraint satisfied $\forall t > 0$

  (Fokas-Sung 1999, Molinet-Saut-Tzvetkov 2007)
  Ablowitz-Villaroel 1991
- Schwartzian initial data in general lead to algebraic fall off of the solution



Dispersionless KP

T. Grava, C. Klein and J. Eggers, *Shock formation in the dispersionless Kadomtsev-Petviashvili equation*, arXiv:1505.06453

- dispersionless KP equation (Kohklov-Zabolotskaya equation)

$$(u_t + uu_x)_x = \pm u_{yy}$$

- characteristics of the Hopf equation (see also Manakov-Santini)

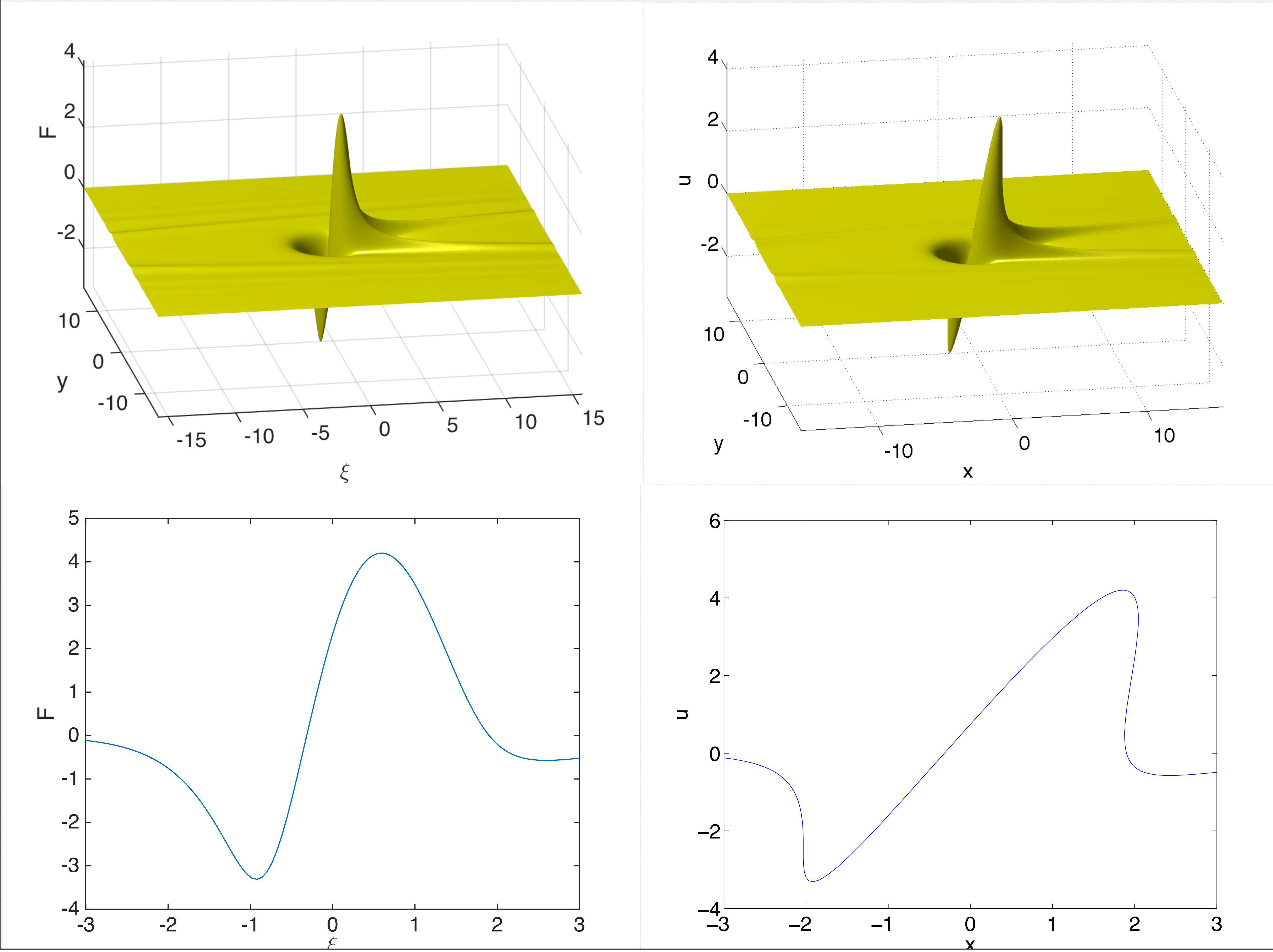
$$\begin{cases} u(x,y,t) = F(\xi,y,t) \\ x = tF(\xi,y,t) + \xi \end{cases}$$

- transformed equation

$$F_t = \partial_\xi^{-1} F_{yy} + t(F_\xi \partial_\xi^{-1} F_{yy} - F_y^2), \quad F(\xi,y,0) = u(x,y,0)$$

numerics: longer existence time of smooth solution than for dKP solution

$$u_0(x,y) = -6\partial_x \mathrm{sech}^2(x)$$



Conjecture: break-up in KP

B. Dubrovin, T. Grava, and C. Klein, On critical behaviour in generalized Kadomtsev–Petviashvili equations, arXiv:1510.01580

$$u(x,y,t;\epsilon) = u_c + \frac{6}{nu_c^{n-1}}\left(\frac{\epsilon^2}{\kappa^2}\right)^{\frac{1}{7}} U\left(\frac{X}{(\kappa\epsilon^6)^{1/7}}, \frac{T}{(\kappa^3\epsilon^4)^{1/7}}\right) + \bar{y}(F_y - F_\xi \frac{G_{\xi\xi y}}{G_{\xi\xi\xi}}) + O(\epsilon^{\frac{4}{7}})$$

$$\kappa = -36\, G^c_{\xi\xi\xi} t_c^4, \quad G(\xi,y,t) := F^n(\xi,y,t)$$

$$\bar{x} - \bar{t}(G^c + t_c G^c_t) - \bar{t}\bar{y}(G^c_y + t_c G^c_{yt}) - t_c(G^c_y \bar{y} + \frac{1}{6}G^c_{yyy}\bar{y}^3 + \frac{1}{2}G^c_{yy}\bar{y}^2)$$
$$= \frac{t_c}{6}G^c_{\xi\xi\xi}\bar{\xi}^3 + \frac{1}{2}t_c G^c_{\xi\xi y}\bar{y}\bar{\xi}^2 + \frac{1}{2}(t_c\bar{y}^2 G^c_{\xi yy} + (2t_c G^c_{\xi t} + 2G^c_\xi)\bar{t})\bar{\xi} + o(\bar{t}^2, \bar{y}^4, \bar{\xi}^4, \bar{t}(\bar{y}^2 + \bar{\xi}^2))$$

$$\zeta = G^c_\xi \left(\bar{\xi} + \frac{G^c_{\xi\xi y}}{G^c_{\xi\xi\xi}}\bar{y}\right)$$

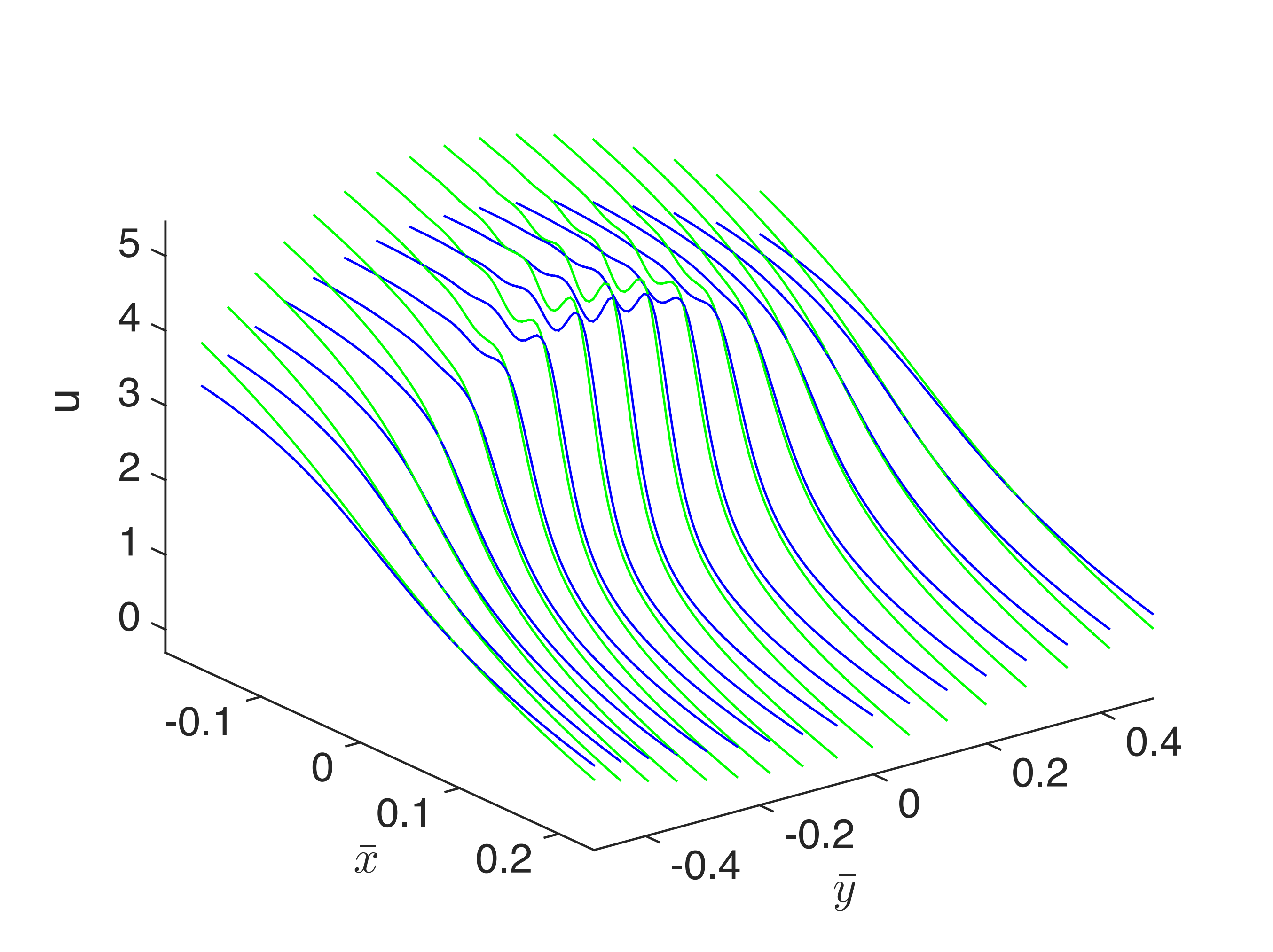
$$X = \left[\bar{x} - \bar{t}(G^c + t_c G^c_t) - \bar{t}\bar{y}(G^c_y + t_c G^c_{yt}) - t_c\left(G^c_y\bar{y} + \frac{1}{2}G^c_{yy}\bar{y}^2 + \frac{1}{6}G^c_{yyy}\bar{y}^3\right)\right.$$
$$\left. -\frac{1}{3}t_c\frac{(G^c_{\xi\xi y})^3}{(G^c_{\xi\xi\xi})^2}\bar{y}^3 + \frac{1}{2}t_c\frac{G^c_{\xi\xi y}G^c_{\xi yy}}{G^c_{\xi\xi\xi}}\bar{y}^3 + G^c_\xi\frac{G^c_{\xi\xi y}}{G^c_{\xi\xi\xi}}\bar{y}\bar{t}\right]$$

$$T = \left[\bar{t} + \frac{t_c^2}{2}\bar{y}^2\left(\frac{(G^c_{\xi\xi y})^2}{G^c_{\xi\xi\xi}} - G^c_{\xi yy}\right)\right],$$

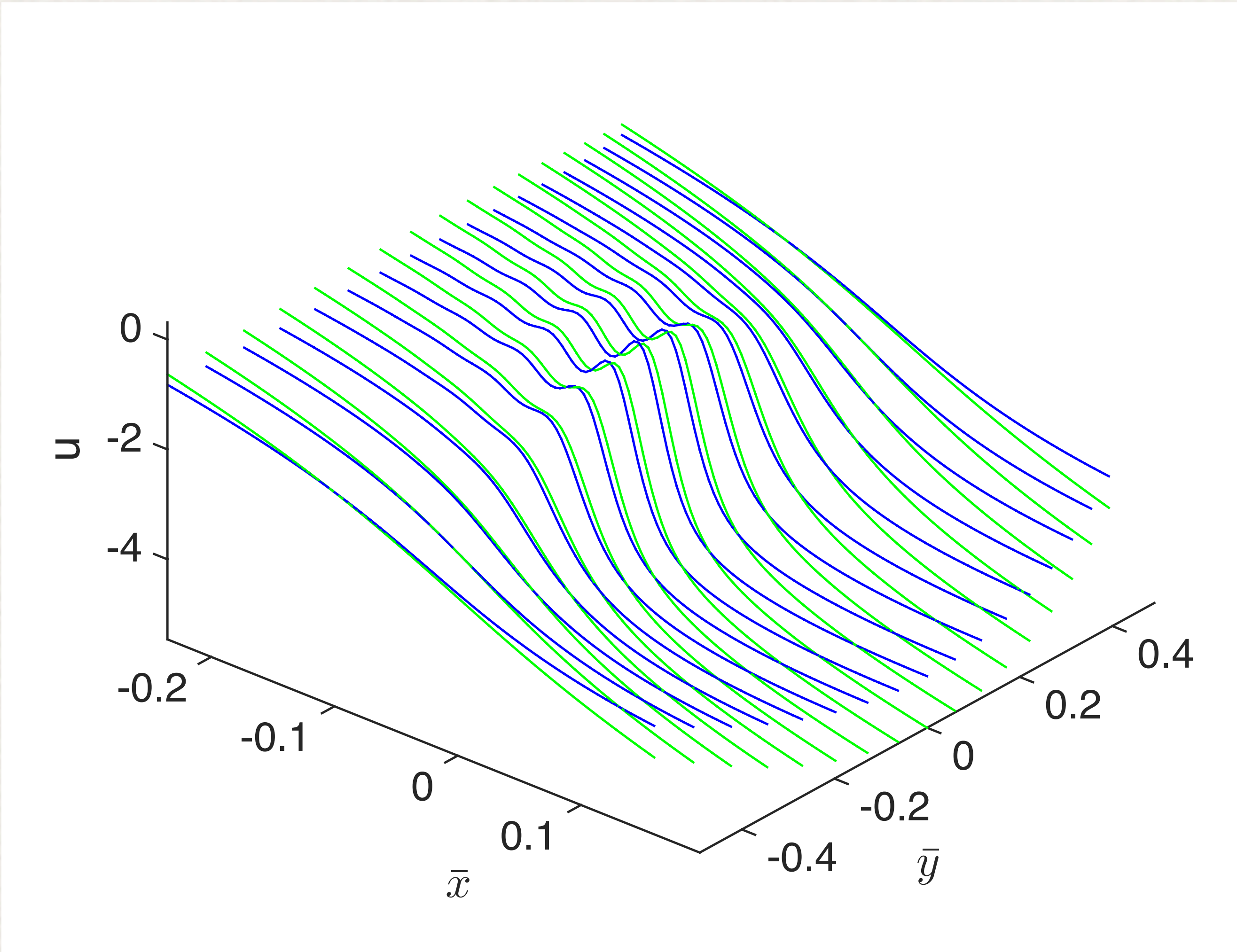
Critical time

$$u_0(x,y) = -6\partial_x \mathrm{sech}^2(x)$$

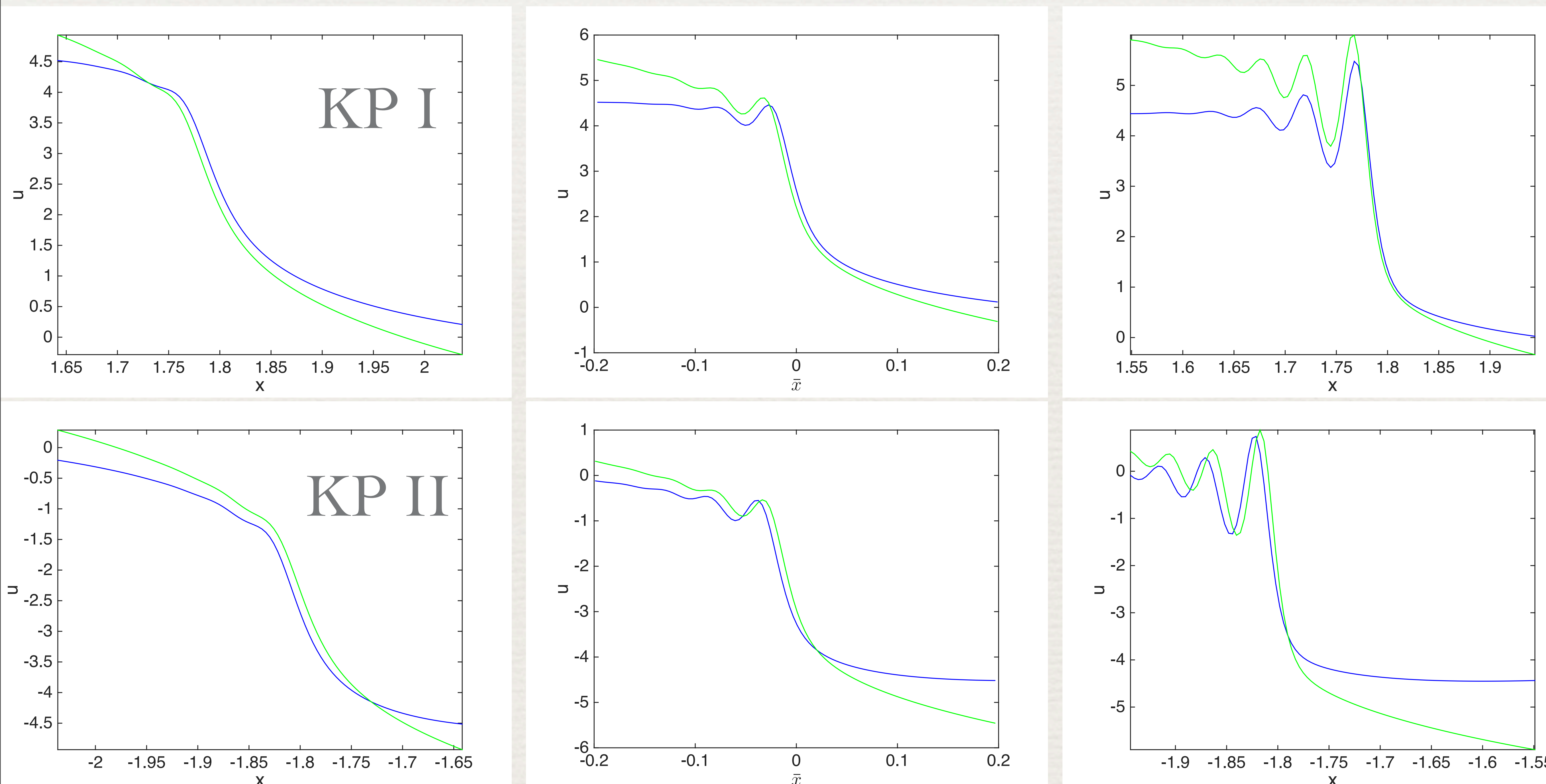
KP I



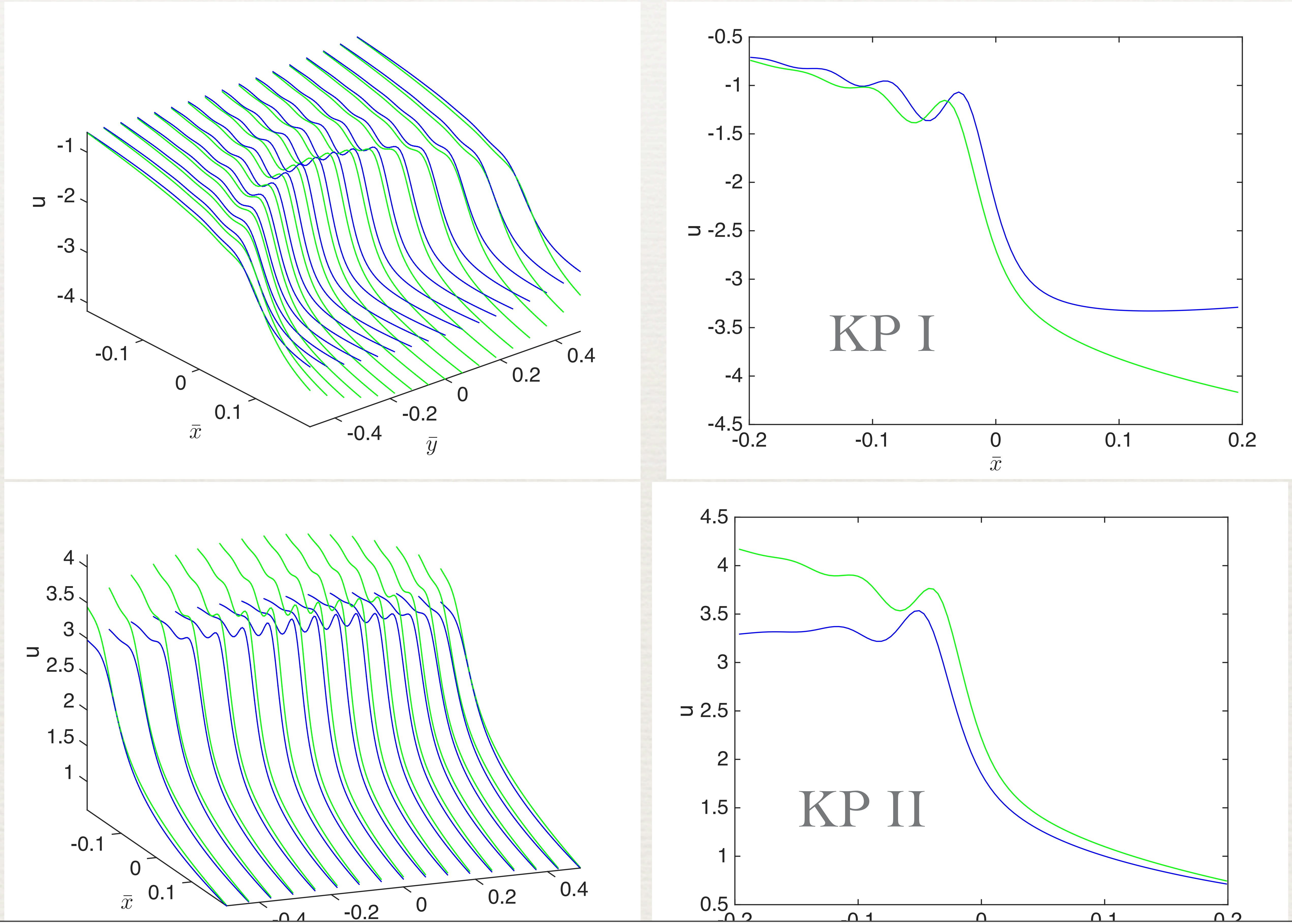
KP II



time dependence

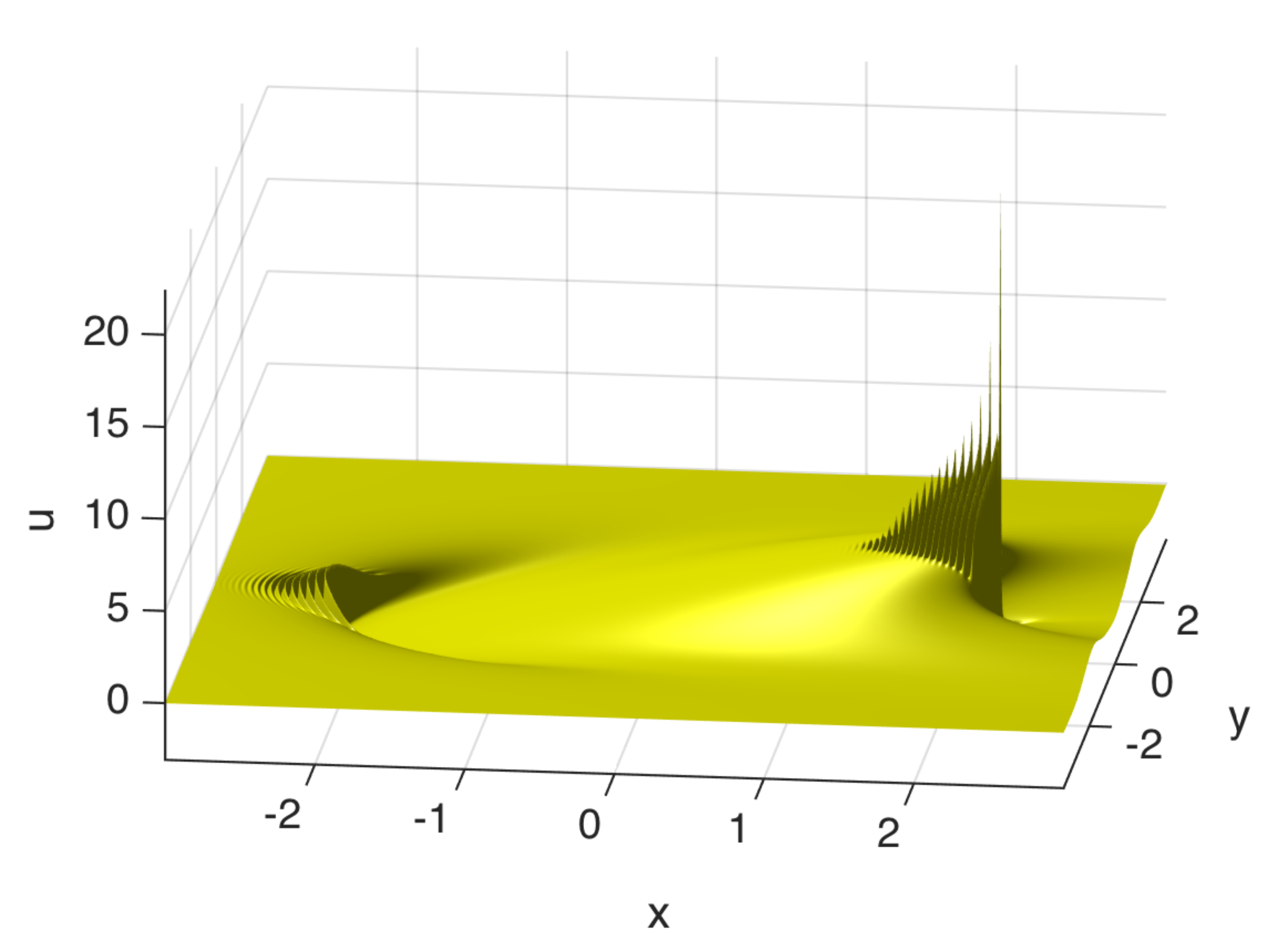


Second break-up

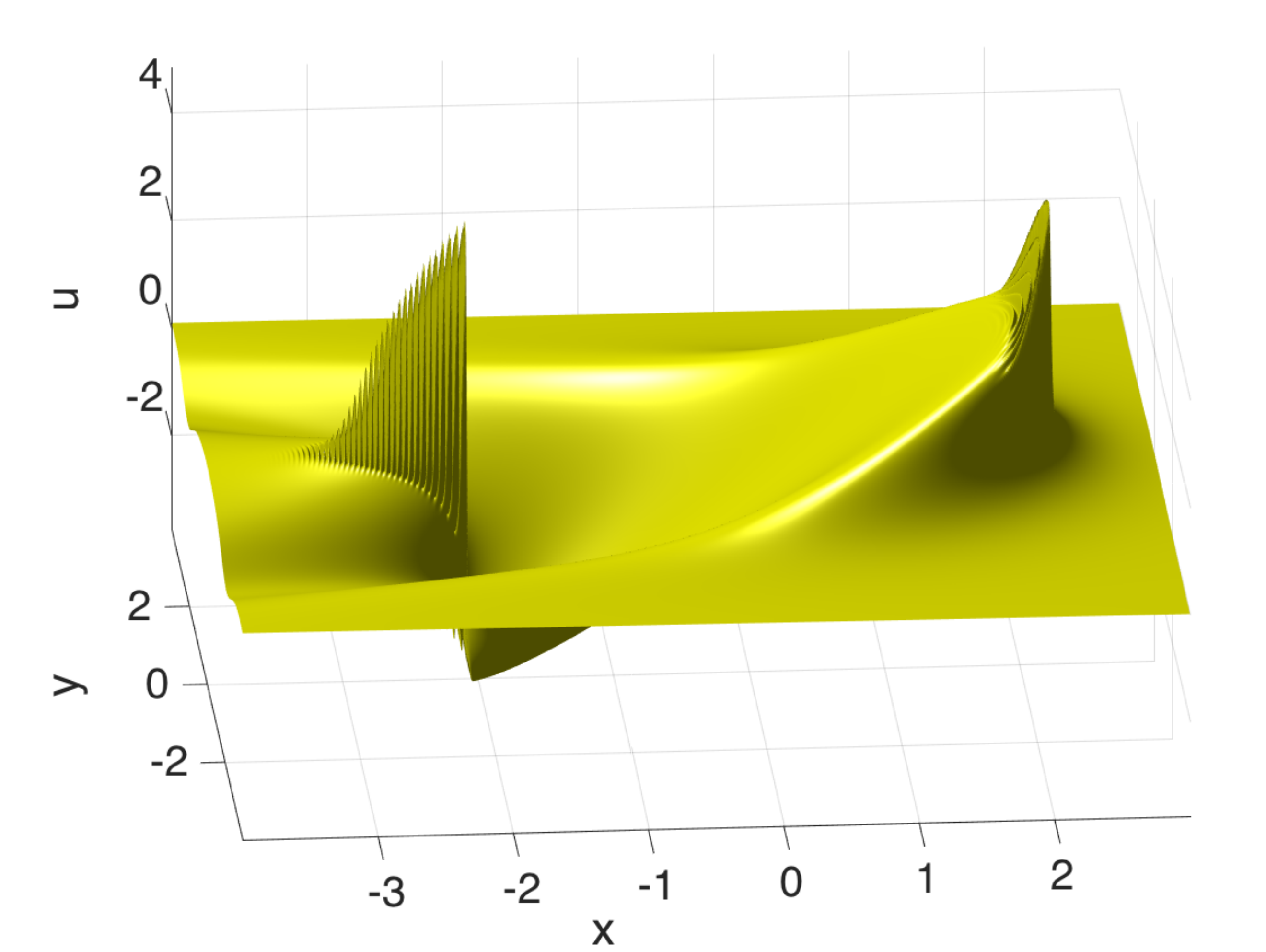


$$t \gg t_c$$

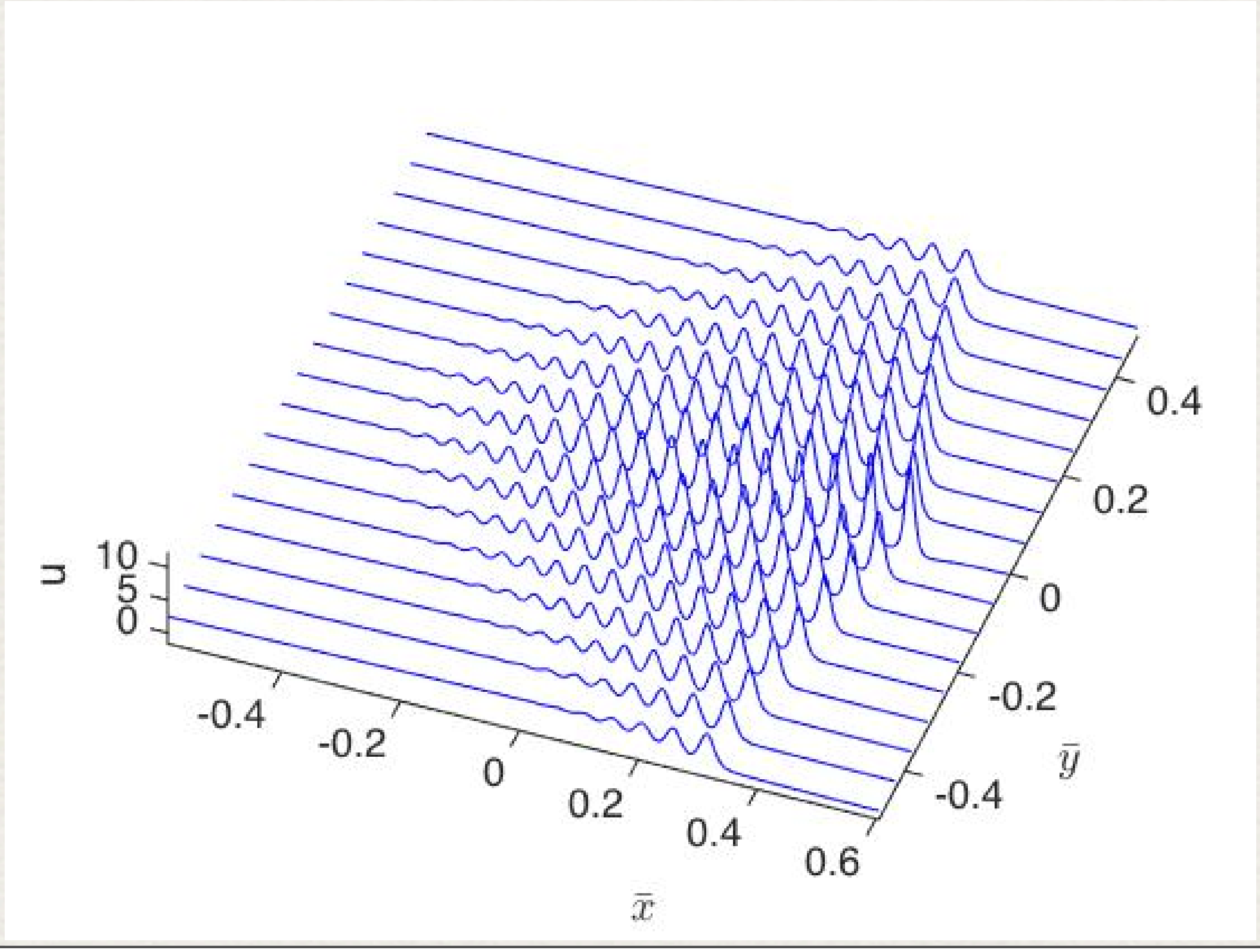
KP I

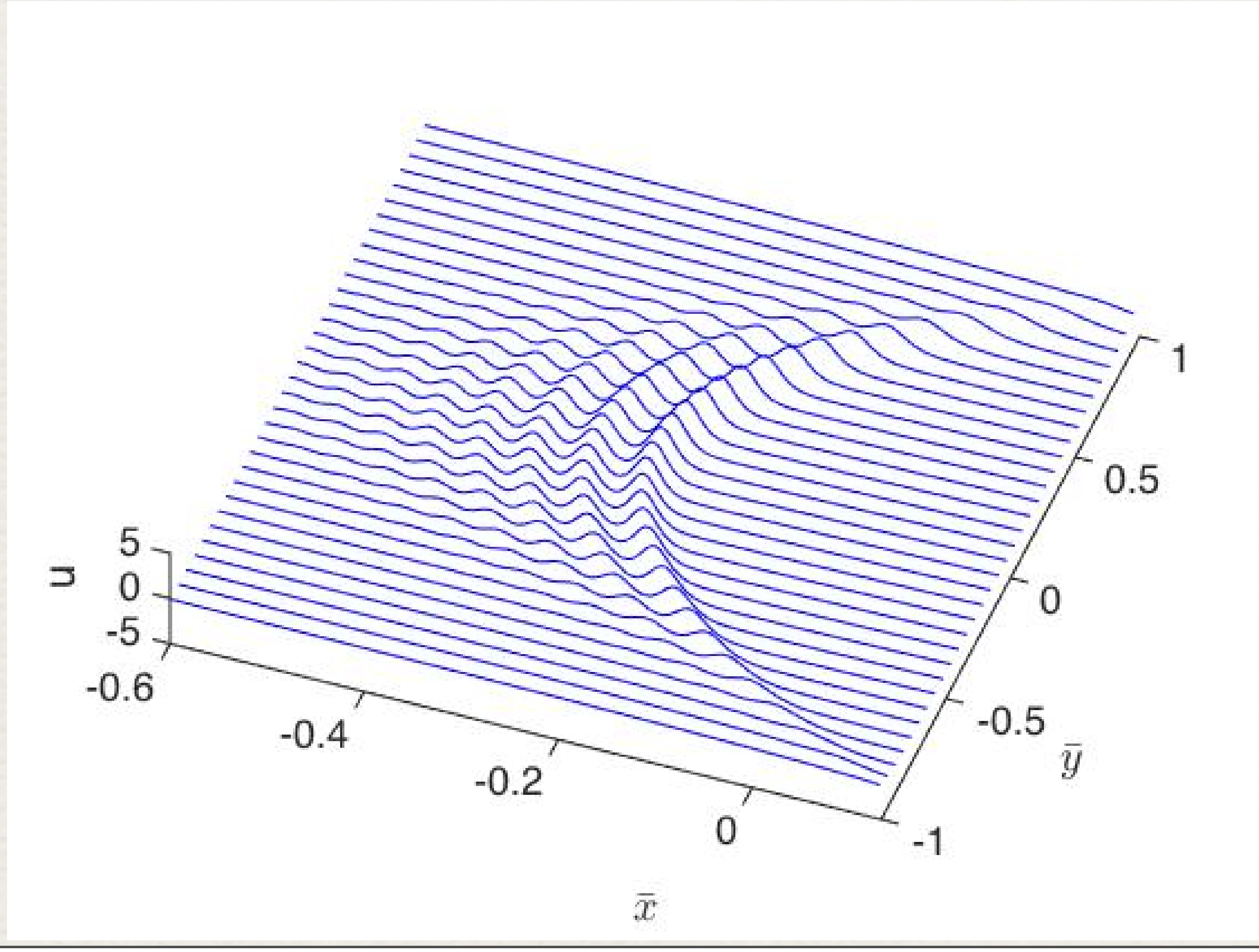


KP II



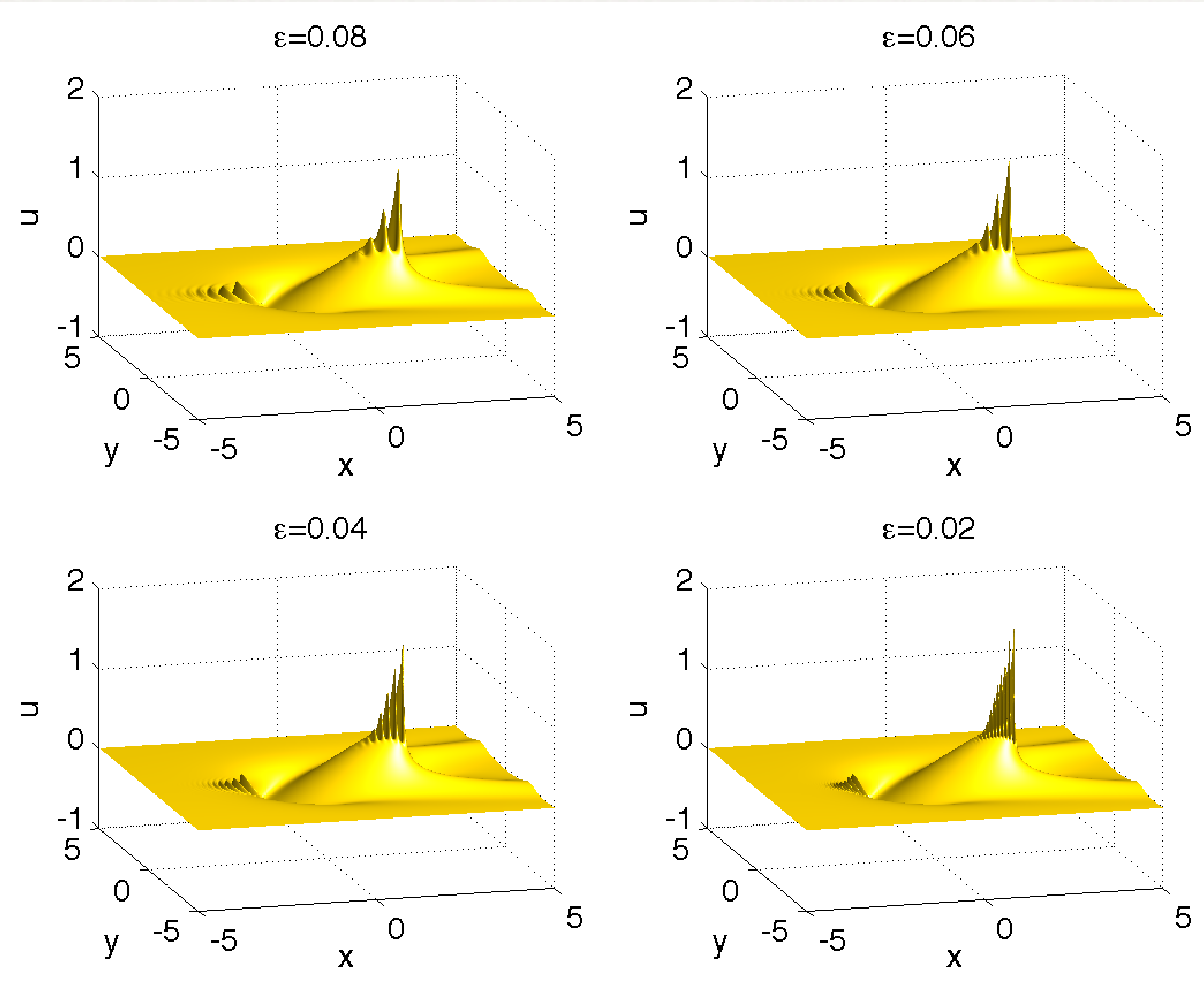
Oscillatory zones in KP I





KP I

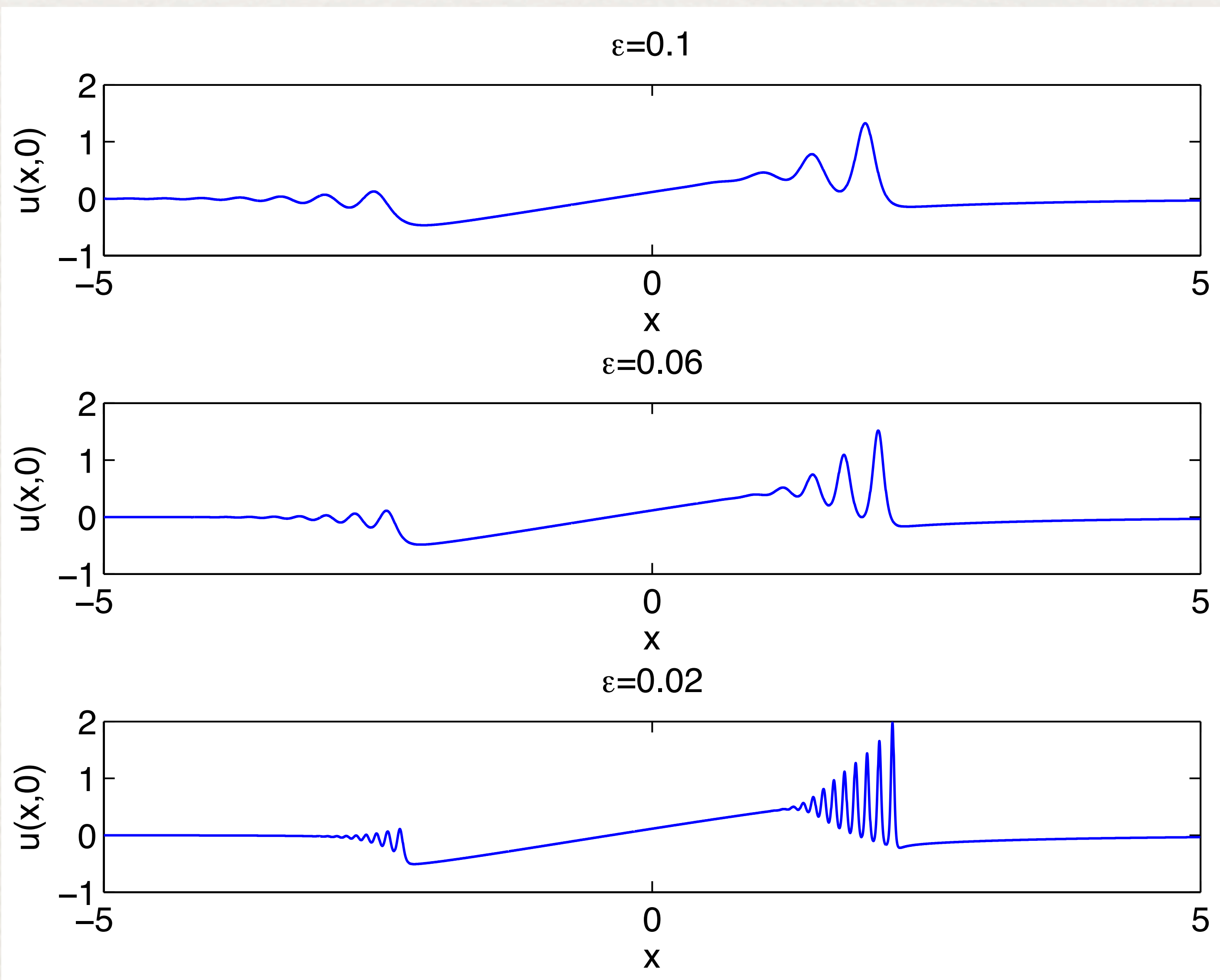
C. Klein and K. Roidot, *Numerical study of shock formation in the dispersionless Kadomtsev-Petviashvili equation and dispersive regularizations*, Physica D, Vol. 265, 1–25, 10.1016/j.physd.2013.09.005 (2013).



$$u(x,y,0) = -\partial_x \operatorname{sech}^2(R),$$
$$R = \sqrt{x^2+y^2}$$
$$\lambda = -1$$

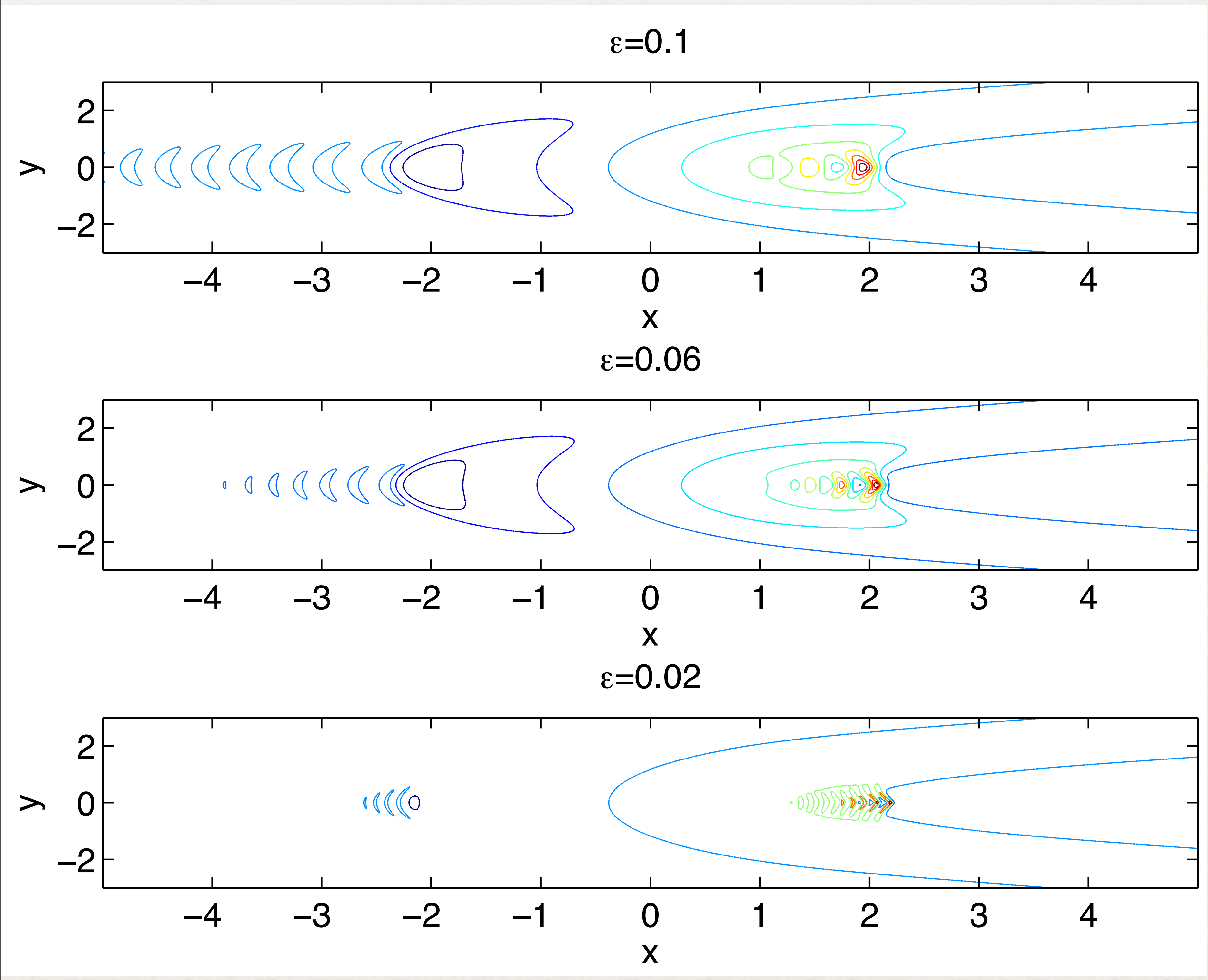
KP I

- same scaling in ϵ of the difference between KP solution and dKP solution at the critical point: $\epsilon^{2/7}$
- dispersive shock for $t \gg t_c$



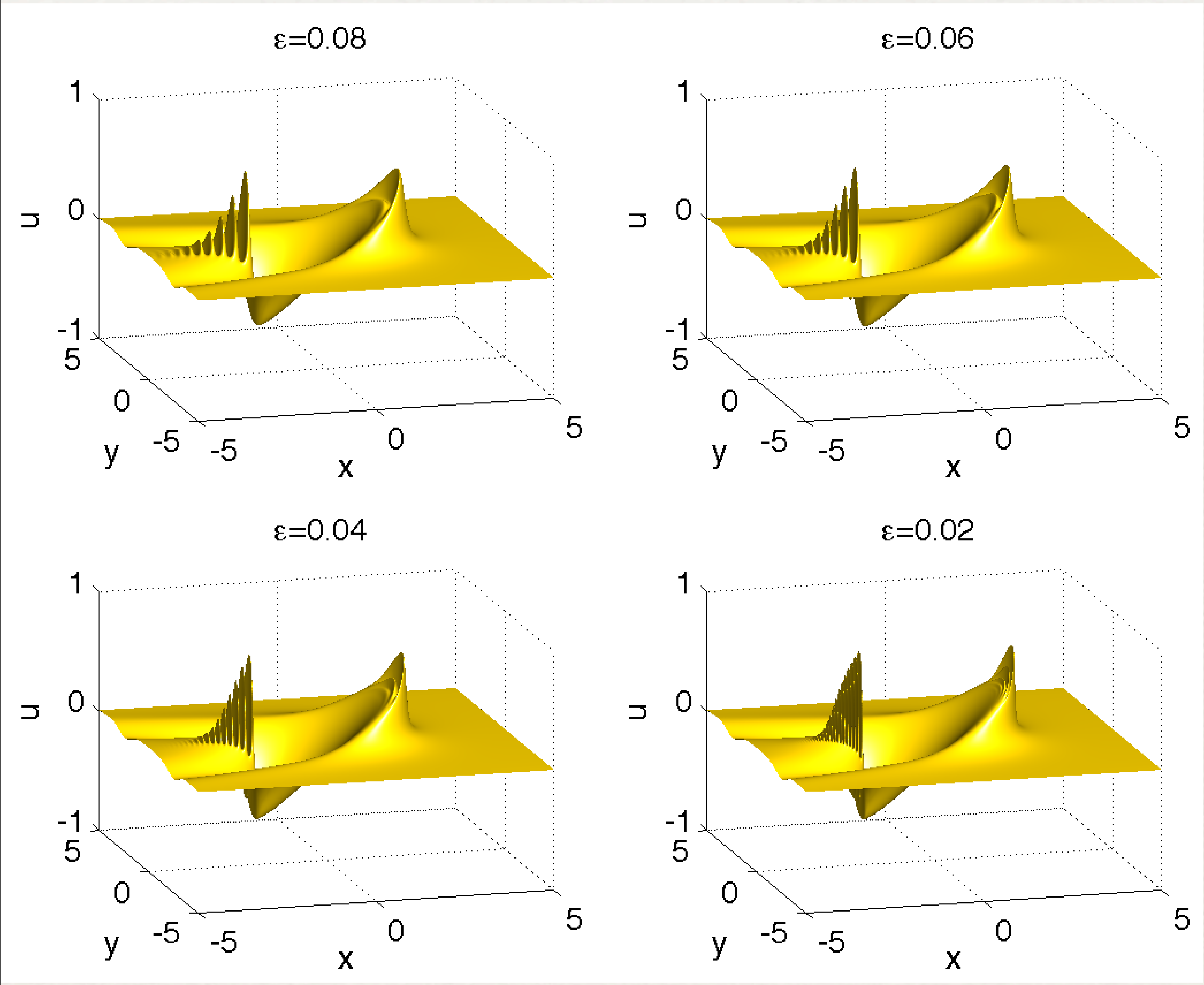
$$\begin{aligned} u(x,y,0) &= -\partial_x \operatorname{sech}^2(R), \\ R &= \sqrt{x^2+y^2} \\ \lambda &= -1 \end{aligned}$$

Contour plot



$$
\begin{aligned}
u(x,y,0) &= -\partial_x \operatorname{sech}^2(R), \\
R &= \sqrt{x^2+y^2} \\
\lambda &= -1
\end{aligned}
$$

KP II



$$
\begin{aligned}
u(x,y,0) &= -\partial_x \operatorname{sech}^2(R), \\
R &= \sqrt{x^2+y^2} \\
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\end{aligned}
$$

Generalized Kadomtsev-Petviashvili equations

- generalized Kadomtsev-Petviashvili (gKP) equation, $\lambda = -1$ gKP I, $\lambda = 1$ gKP II
$$u_t + u^n u_x + u_{xxx} + \lambda \partial_x^{-1} u_{yy} = 0$$
- nonlocal equation, algebraic decrease towards infinity of the solution even for rapidly decreasing initial data
- constraint
$$\int_{\mathbb{R}} \partial_{yy} u(x,y,t)\, dx = 0, \quad \forall t > 0$$
if not satisfied by the initial condition, solution not regular in t
- numerical study of blow-up by Wang, Ablowitz, Segur (1994)
- gKP I solitons (de Bouard, Saut 1997), unstable for $n \geq 4/3$
$$-cQ_{zz} + \frac{1}{n+1}(Q^{n+1})_{zz} + Q_{zzzz} + \lambda Q_{yy} = 0$$

Dynamic rescaling

C. Klein and R. Peter, *Numerical study of blow-up in solutions to generalized Kadomtsev-Petviashvili equations*, Discr. Cont. Dyn. Syst. B 19(6), (2014) doi:10.3934/dcdsb.2014.19.1689

- coordinate change

$$\xi = \frac{x - x_m}{L}, \quad \eta = \frac{y - y_m}{L^2}, \quad \frac{d\tau}{dt} = \frac{1}{L^3}, \quad U = L^{2/n} u$$

$||u||_2$ invariant for $n = 4/3$

- rescaled equation

$$U_\tau - a\left(\frac{2}{n}U + \xi U_\xi + 2\eta U_\eta\right) - v_\xi U_\xi - v_\eta U_\eta + U^n U_\xi + U_{\xi\xi\xi} + \lambda \int_{-\infty}^{\xi} U_{\eta\eta}\, d\xi = 0$$

- blow-up

$$-a^\infty\left(\frac{2}{n}U^\infty + \xi U_\xi^\infty + 2\eta U_\eta^\infty\right) - v_\xi^\infty U_\xi^\infty - v_\eta^\infty U_\eta^\infty + (U^\infty)^n U_\xi^\infty + \epsilon^2 U_{\xi\xi\xi}^\infty + \lambda \int_{-\infty}^{\xi} U_{\eta\eta}^\infty\, d\xi = 0$$

- numerical instabilities due to algebraic decay of the solutions

gKP I, critical case

$$n = 4/3, \quad u_0 = 12\partial_{xx}\exp(-x^2-y^2)$$

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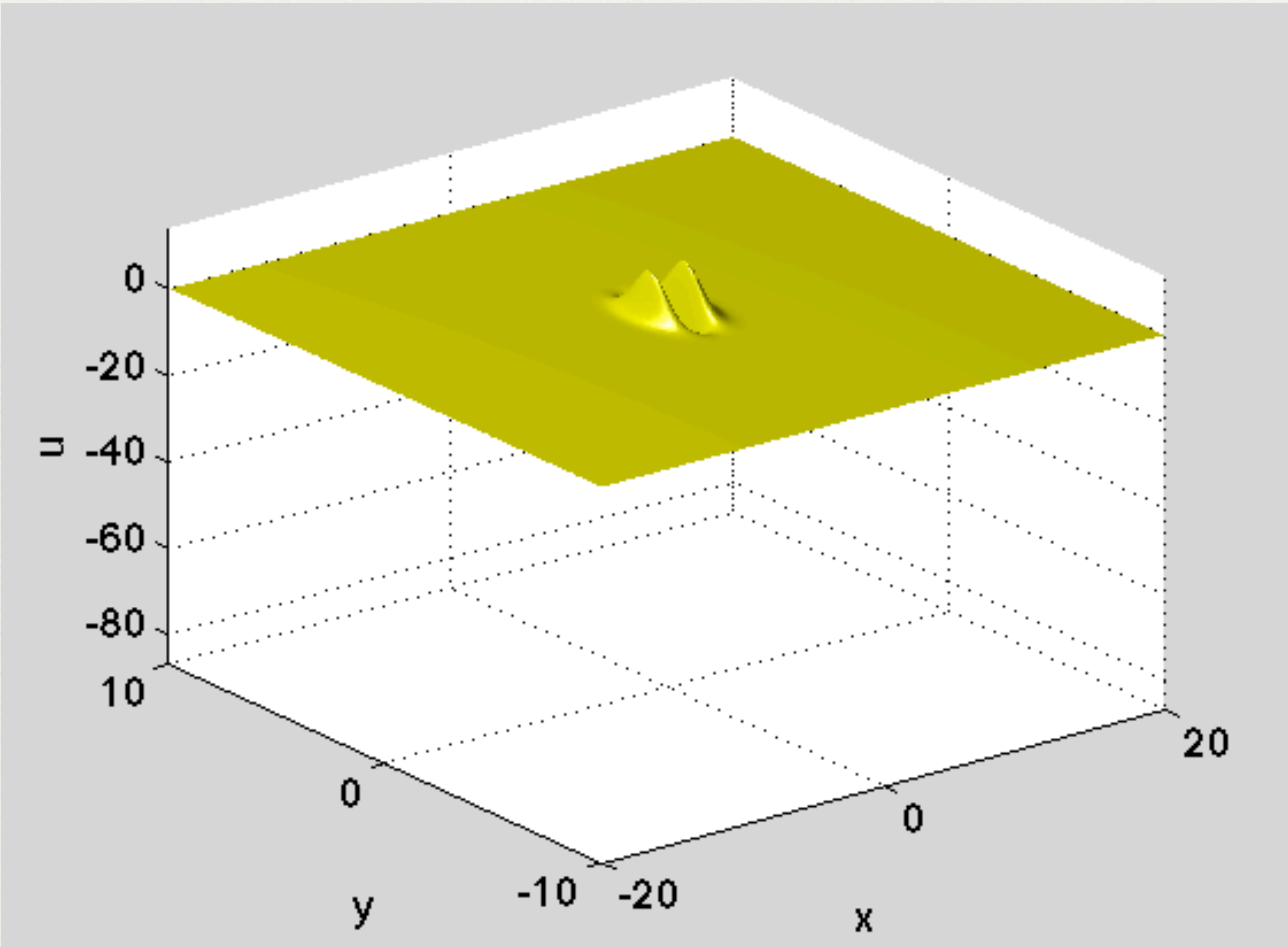
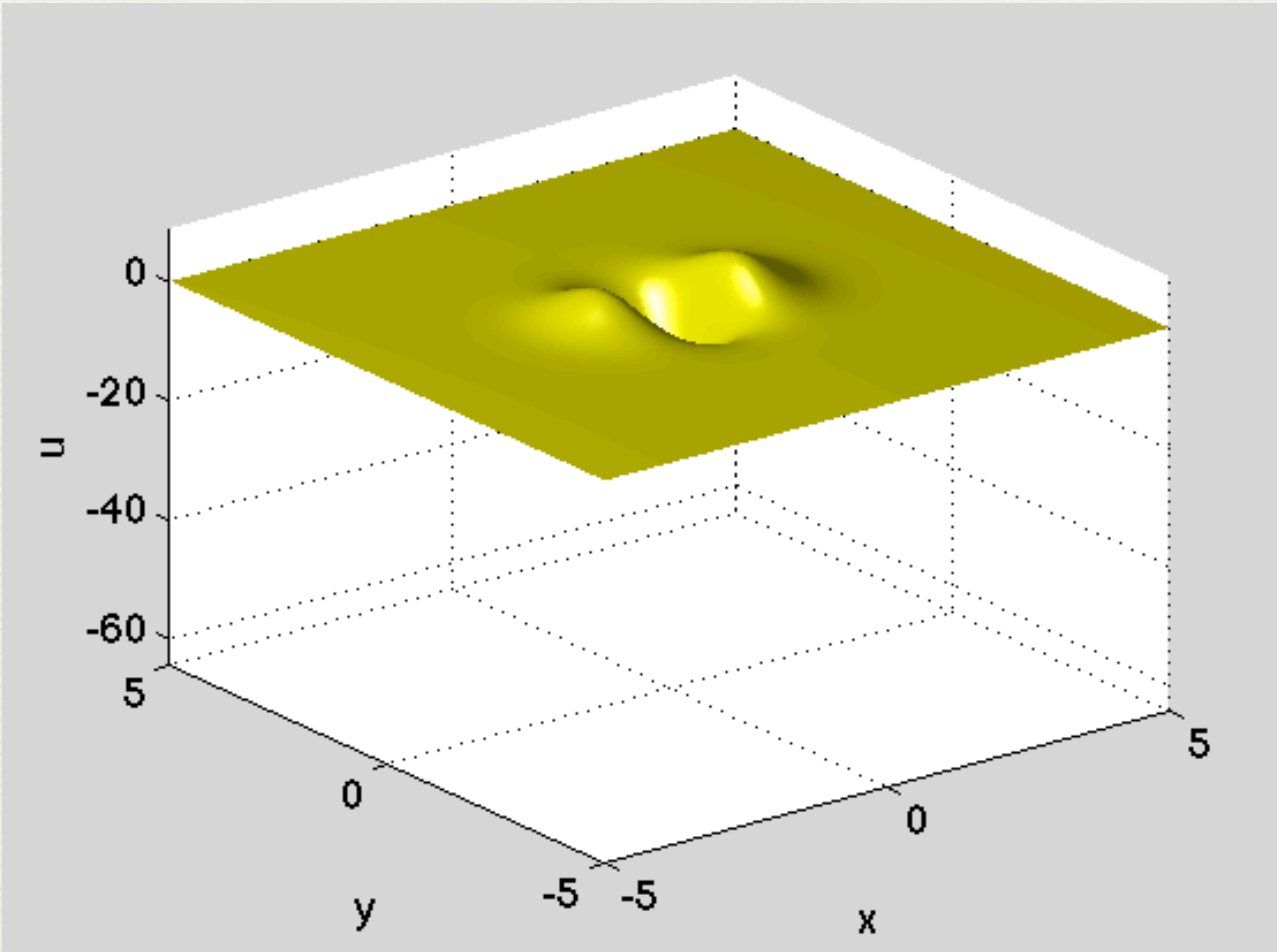


gKP I, supercritical case

$$n = 2, \quad u_0 = 6\partial_{xx}\exp(-x^2 - y^2)$$

gKP I, supercritical case

$$n = 2, \quad u_0 = 6\partial_{xx}\exp(-x^2 - y^2)$$



Conjecture

- for $n < 4/3$, the solution is smooth for all t.
- for gKP II, the solution is smooth for all t for $n \leq 2$.
- for gKP I with $n = 4/3$, initial data with sufficiently small energy and sufficiently large mass lead to blow-up at $t^* < \infty$; asymptotically for $t \sim t^*$, the solution is given by a rescaled soliton where the scaling factor $L \propto 1/\tau$ for $\tau \to \infty$. This implies the blow-up is characterized by

$$||u||_\infty \propto \frac{1}{(t^* - t)^{3/4}}, \quad ||u_y||_2 \propto \frac{1}{t^* - t}. \tag{1}$$

- for gKP I with $n > 4/3$ and gKP II with $n > 2$, initial data with sufficiently small energy and sufficiently large mass lead to blow-up at $t^* < \infty$; asymptotically for $t \sim t^*$, the solution is given by a localized solution to the asymptotic PDE, which is conjectured to exist and to be unique, after rescaling where the scaling factor $L \propto \exp(\kappa\tau)$ for $\tau \to \infty$ with κ a negative constant. This implies the blow-up is characterized by

$$||u||_\infty \propto \frac{1}{(t^* - t)^{2/(3n)}}, \quad ||u_y||_2 \propto \frac{1}{(t^* - t)^{(1+4/n)/6}}. \tag{2}$$

Universality conjecture: hyperbolic case

Behaviour of the solution near the critical point (x_c, t_c) when $x_\pm = x - x^c \pm \lambda^c_\pm (t - t_c) \to 0$ in the double scaling limit when $\epsilon \to 0$

$$r_-(x,t,\epsilon) = r_-(x_c,t_c) + \alpha\epsilon^{\frac{2}{7}} U\left(\frac{x_-}{\beta\epsilon^{\frac{6}{7}}}, \frac{x_+}{\gamma\epsilon^{\frac{4}{7}}}\right) + O(\epsilon^{\frac{4}{7}})$$

$$r_+(x,t,\epsilon) = r_+(x_c,t_c) + \delta\epsilon^{\frac{4}{7}} U''\left(\frac{x_-}{\beta\epsilon^{\frac{6}{7}}}, \frac{x_+}{\gamma\epsilon^{\frac{4}{7}}}\right) + O(\epsilon^{\frac{6}{7}}),$$

where α, β, γ and δ are constants and $U(X,T)$ solves the PI-2 equation,

$$X = TU - \left(\frac{1}{6}U^3 + \frac{1}{24}(U_X^2 + 2UU_{XX}) + \frac{1}{240}U_{XXXX}\right).$$

Universality conjecture: elliptic case

Riemann invariants r_+ and r_- are complex conjugate. Near the point of elliptic umbilic catastrophe (x_c, t_c) the local solution to the perturbed Hamiltonian system is approximated in the double scaling limit by

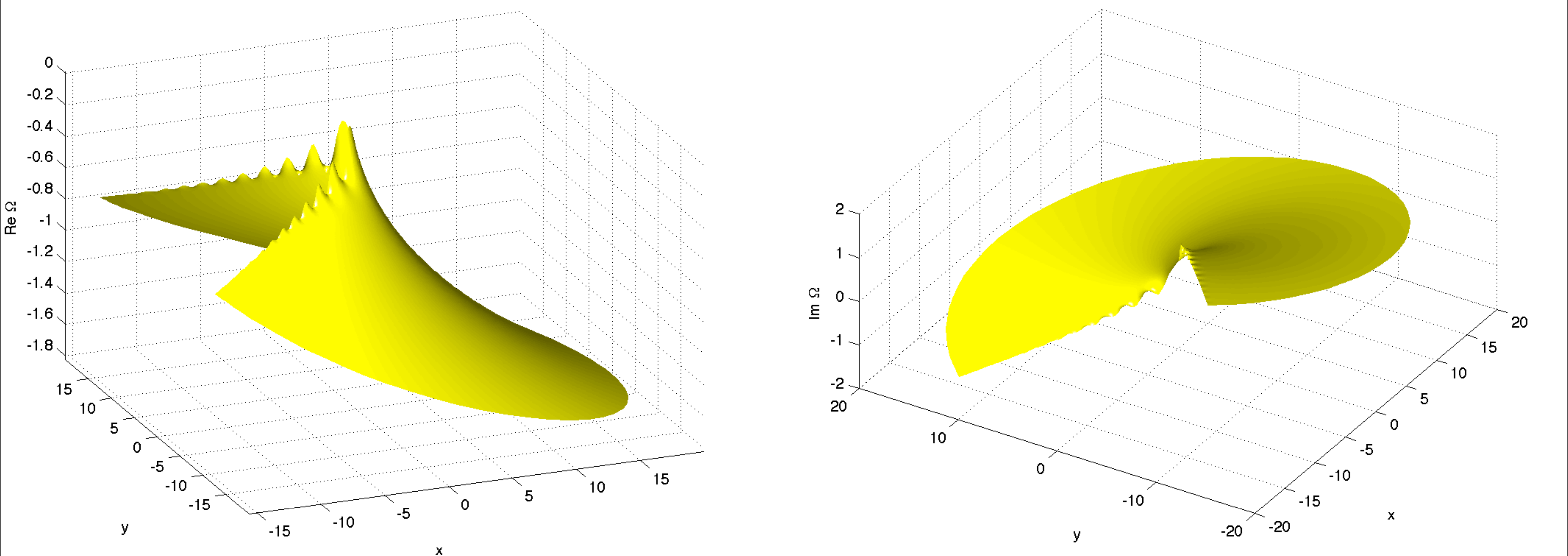
$$r_+(x,t,\epsilon) = r_+(x_c,t_c) + \alpha\epsilon^{\frac{2}{5}}\Omega\left(\frac{x - x_c + \lambda_+^c(t-t_c)}{\gamma\epsilon^{\frac{4}{5}}}\right) + O(\epsilon^{\frac{4}{5}})$$

where α, β, γ are constants and Ω is the tritronquée solution to the Painlevé I equation $\Omega_{\xi\xi} = 6\Omega^2 - \xi$ determined uniquely by the asymptotic conditions

$$\Omega(\xi) \simeq -\sqrt{\frac{\xi}{6}}, \quad |\xi| \to \infty, \quad |\arg\ \xi| < \frac{4}{5}\pi.$$

Further conjecture: the tritronquée solution is pole-free.

Conjecture: no poles in the sector $|arg(z)|<4\pi/5$



- harmonic function with *tritronquée* boundary data

Defocusing NLS

$$\psi_0(x) = \exp(-x^2),$$

$$\epsilon = 0.5,$$

$$0 \leq t \leq 1,$$

$$u = |\psi|^2$$

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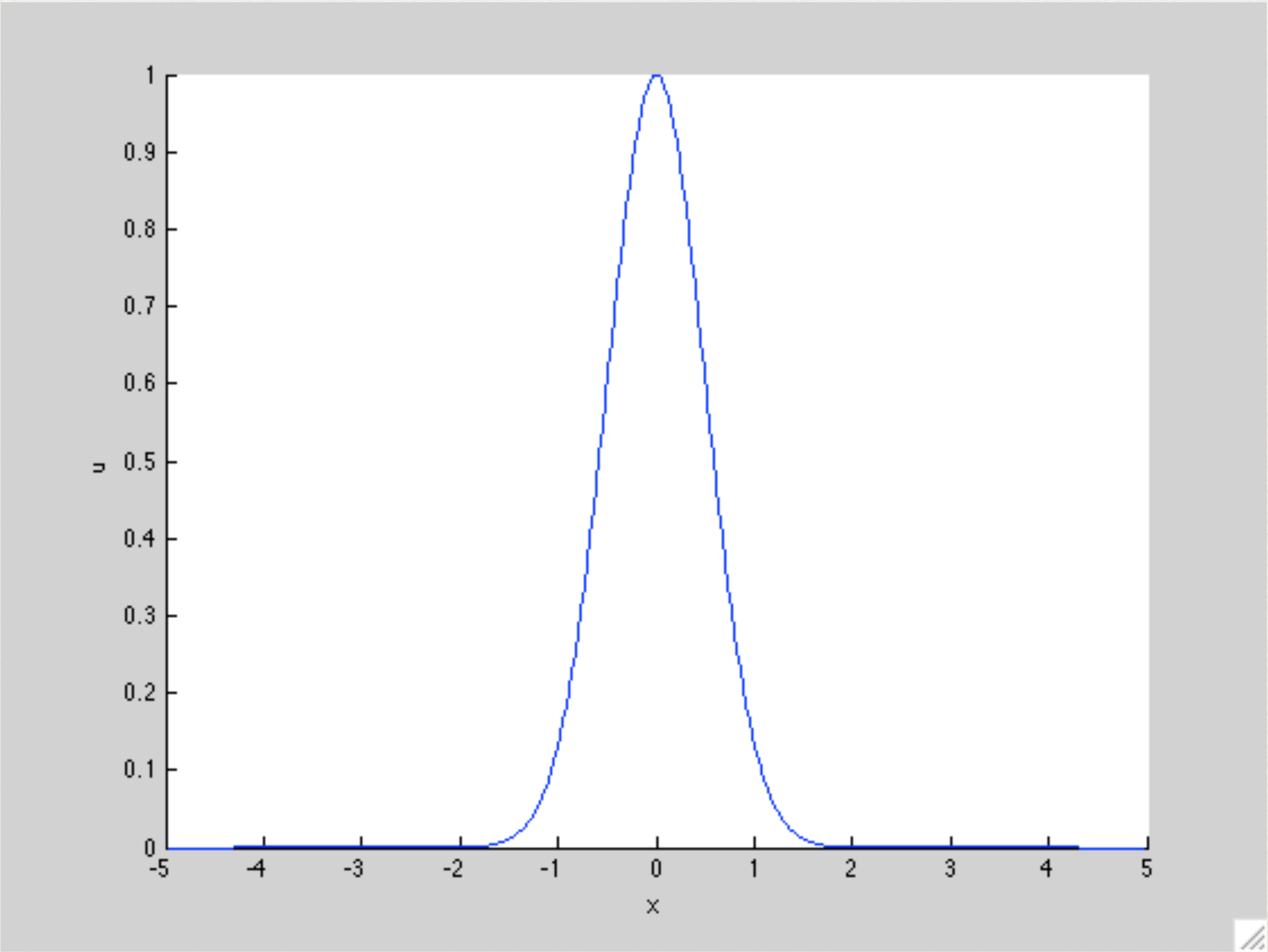


Focusing NLS

$$\psi_0(x) = \exp(-x^2),$$
$$\epsilon = 0.1,$$
$$0 \leq t \leq 0.8,$$
$$u = |\psi|^2$$

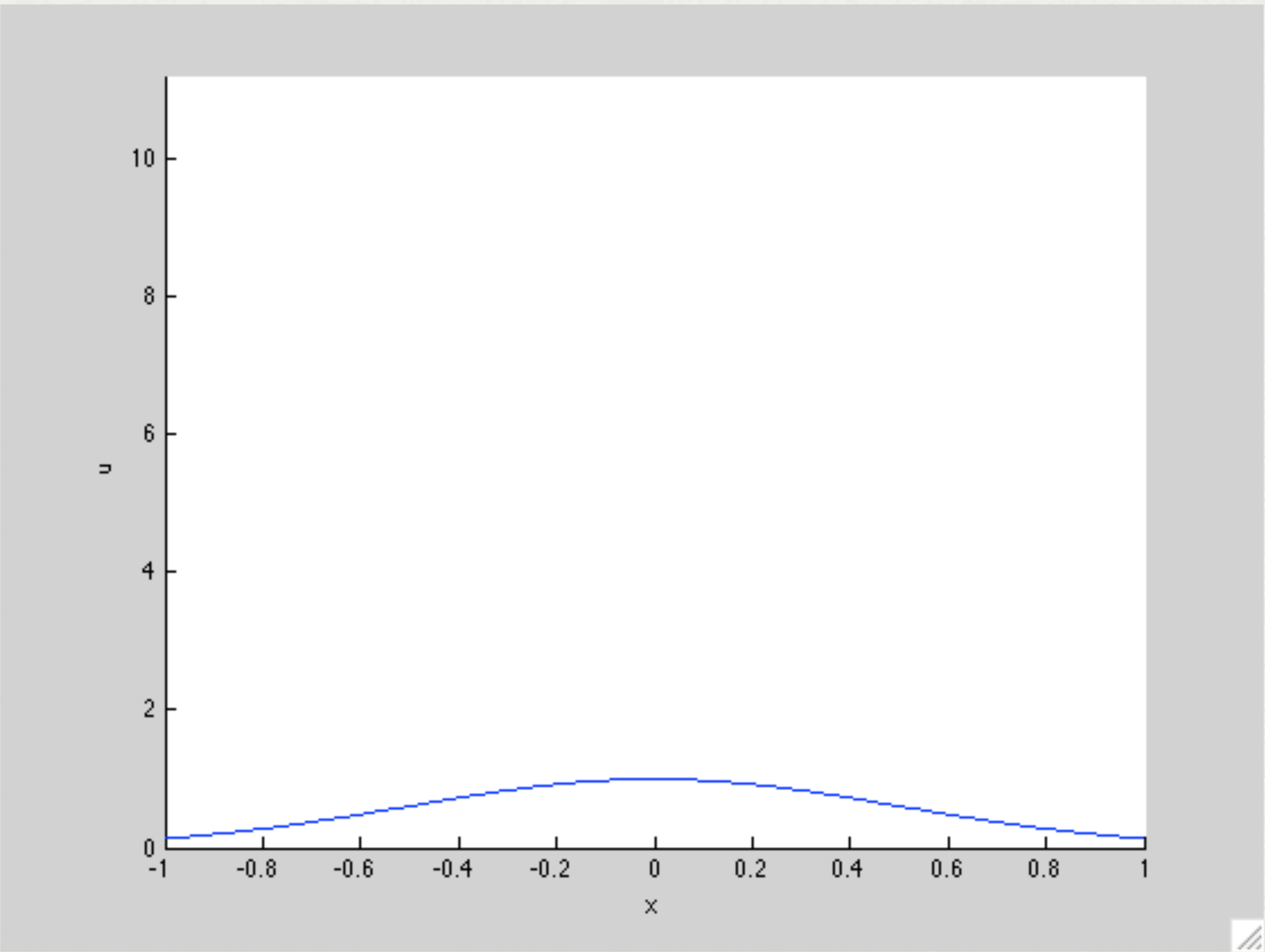
Focusing NLS

$$\psi_0(x) = \exp(-x^2),$$

$$\epsilon = 0.1,$$

$$0 \leq t \leq 0.8,$$

$$u = |\psi|^2$$

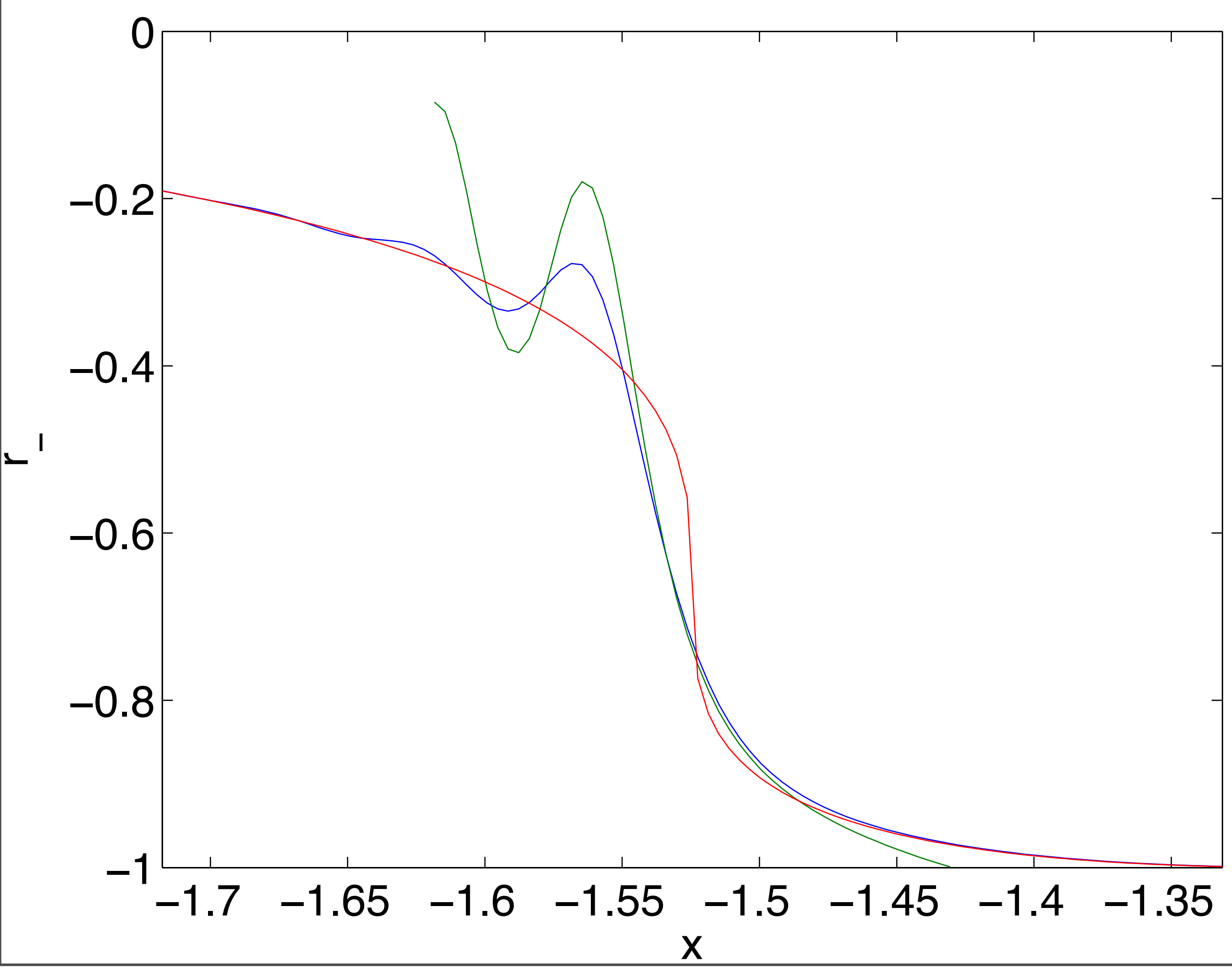


Quintic defocusing NLS

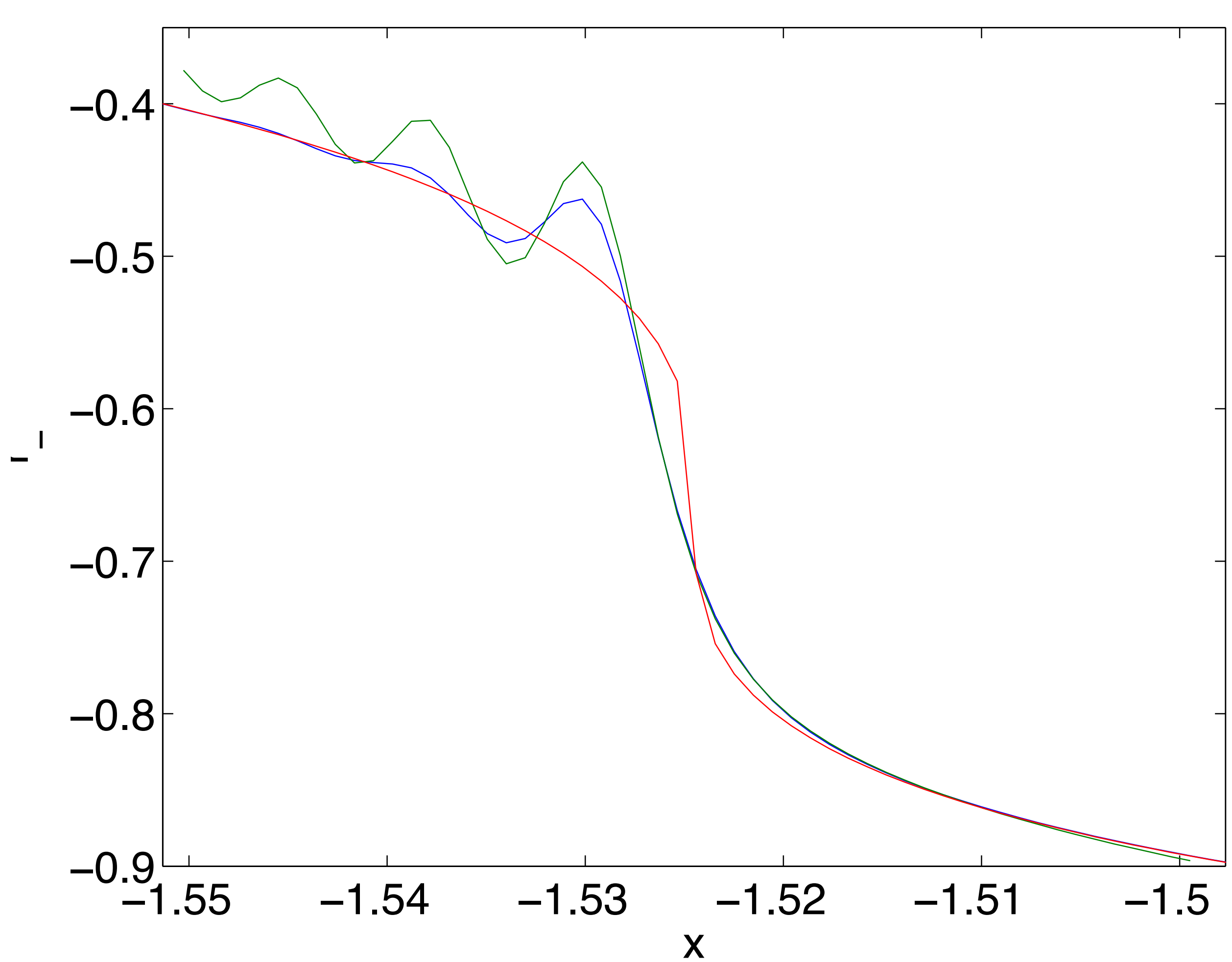
$$u_t + (uv)_x = 0, \quad v_t + vv_x + uu_x + \frac{\epsilon^2}{4}\left(\frac{u_x^2}{2u^2} - \frac{u_{xx}}{u}\right)_x = 0.$$

Riemann invariants $r_\pm = v \pm u$. Initial data $u(x,0) = \mathrm{sech}^2 x$, $v(x,0) = -\tanh^2 x$. Gradient catastrophe at $t_c = 3\sqrt{3}/4$, $x_- = \ln((\sqrt{3}+1)/\sqrt{2}) + \sqrt{3}/2) \sim 1.5245$, $r_-^c = -2/3$.

$\epsilon = 10^{-2}$



$\epsilon = 10^{-3}$

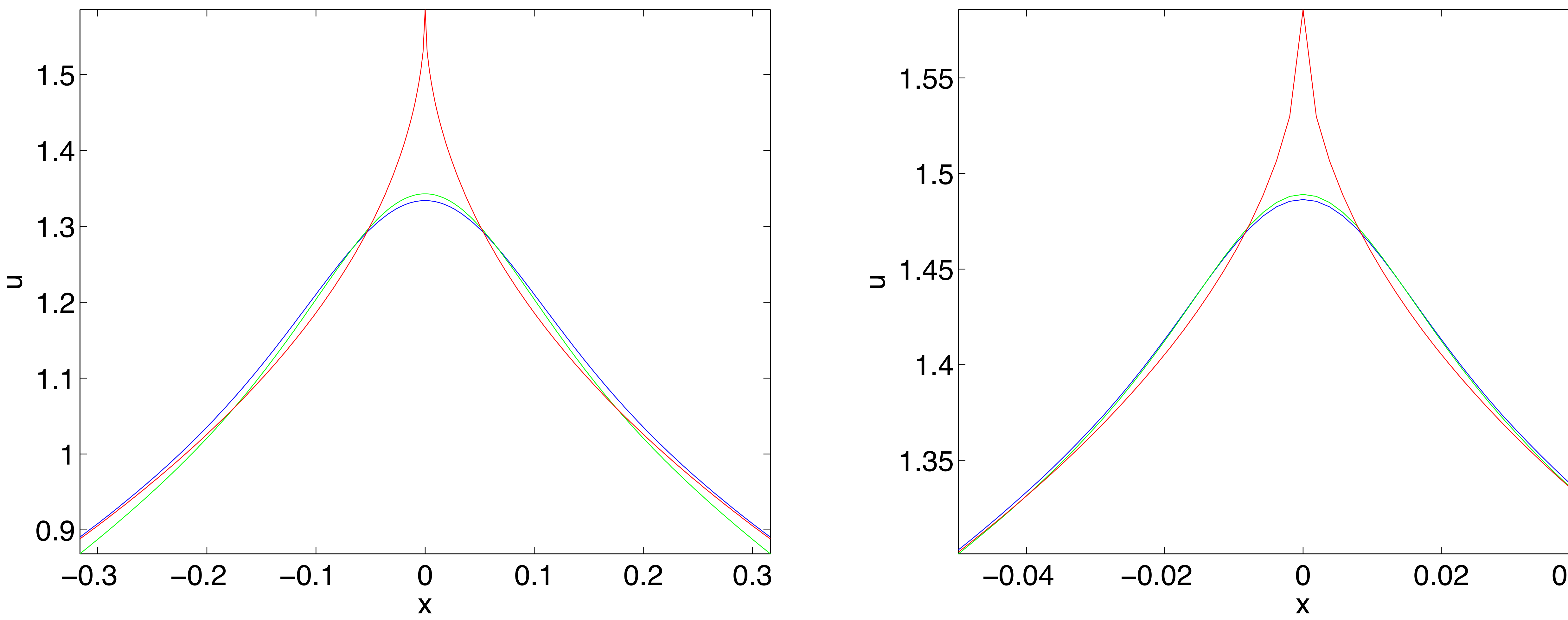


Focusing quintic NLS

B. Dubrovin, T. Grava, C. Klein and A. Moro, *On critical behaviour in systems of Hamiltonian partial differential equations*, J. Nonl. Sci., 25(3), 631–707 (2015).

$$\psi_0 = \mathrm{sech} x$$

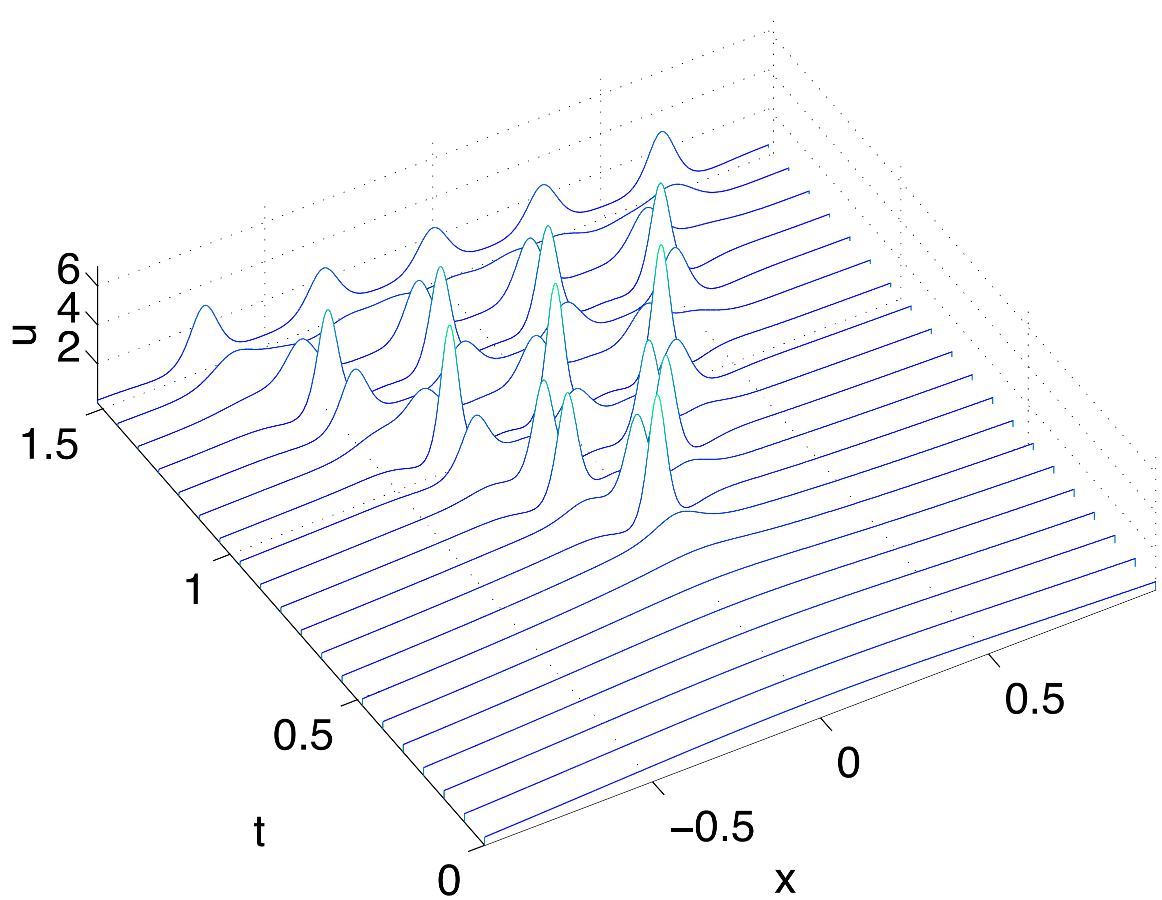
$\epsilon = 0.1$ $\epsilon = 0.01$

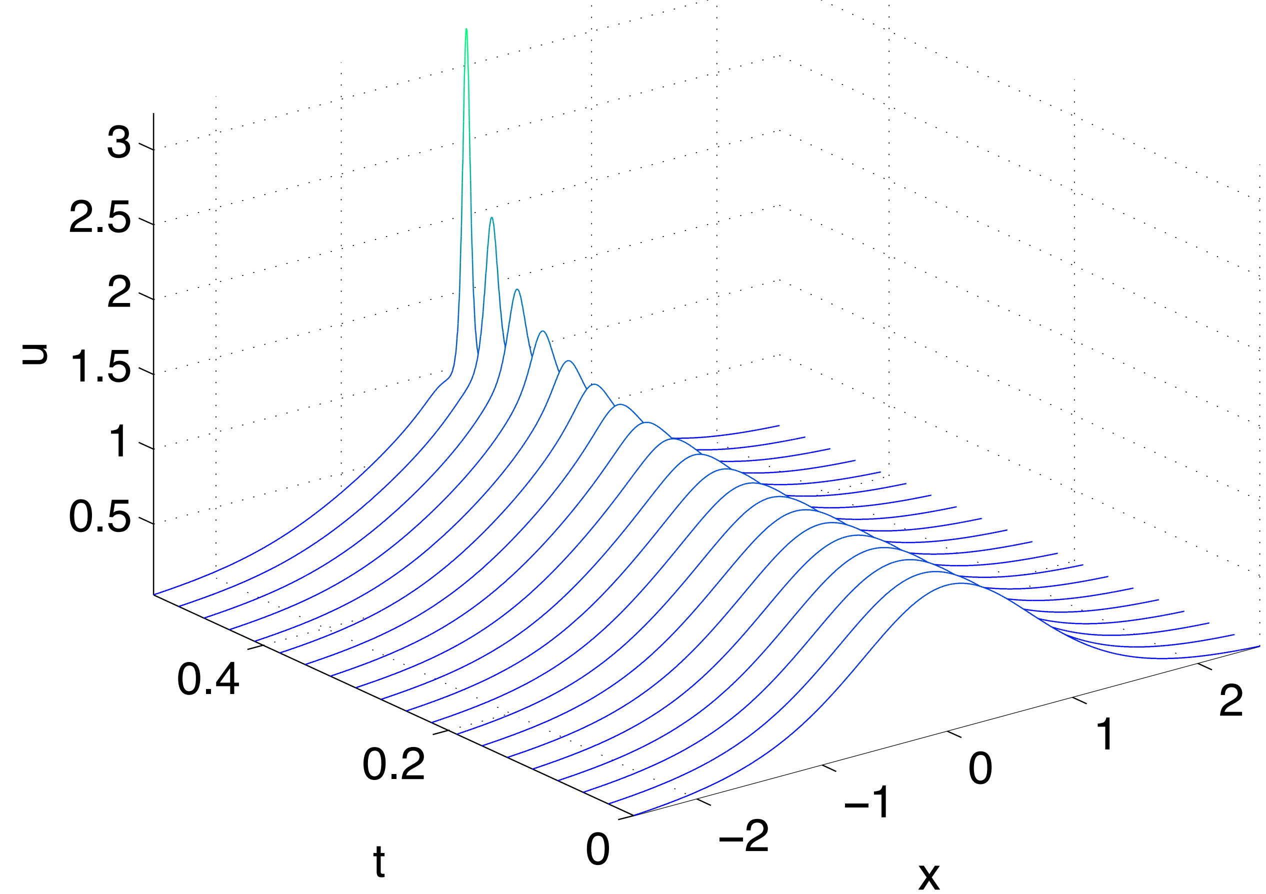


Blow-up

- unstable blow-up: $||\Psi||_\infty \propto 1/(t^* - t)$, stable blow-up: $||\Psi||_\infty \propto 1/\sqrt{t^* - t}$
- $t^* - t^c = 0(\epsilon^{4/5})$
- pole for $\epsilon \to 0$ given by pole of tritronquée solution

cubic NLS $\qquad \psi_0 = \mathrm{sech} x \qquad$ quintic NLS





Davey-Stewartson equation

C. Klein and K. Roidot, *Numerical Study of the semiclassical limit of the Davey-Stewartson II equations*, Nonlinearity 27, 2177-2214 (2014).

- $$\begin{aligned} i\epsilon u_t + \epsilon^2 u_{xx} - \alpha\epsilon^2 u_{yy} + 2\rho\left(\Phi + |u|^2\right)u &= 0 \\ \Phi_{xx} + \alpha\Phi_{yy} + 2\,|u|^2_{xx} &= 0 \end{aligned}$$

- integrable cases: $\alpha = \pm 1$, $\rho = \pm 1$
  - DS I, $\alpha = -1$
  - DS II, hyperbolic-elliptic, $\alpha = 1$

- y-independent potential plus boundary condition at infinity: reduction to NLS

- first numerical studies: White-Weideman (1994), Besse, Mauser, Stimming (2004), McConnell, Fokas, Pelloni (2005)

DS II

- mean field Φ: defocusing ($\rho = -1$) and focusing case ($\rho = 1$) different
- elliptic operator for Φ can be inverted with periodic boundary conditions
- Sung 1995: initial data $\psi_0 \in L^p$, $1 \leq p < 2$ with Fourier transform $\hat{\psi}_0 \in L^1 \cap L^\infty$, smallness condition

$$||\hat{\psi}_0||_{L^1} ||\hat{\psi}_0||_{L^\infty} < \frac{\pi^3}{2} \left(\frac{\sqrt{5}-1}{2} \right)^2$$

  no condition for defocusing case

- initial data $u_0 = \exp(-x^2 - \eta y^2)$: Sung condition

$$\frac{1}{\epsilon^2 \eta} \leq \frac{1}{8} \left(\frac{\sqrt{5}-1}{2} \right)^2 \sim 0.0477.$$

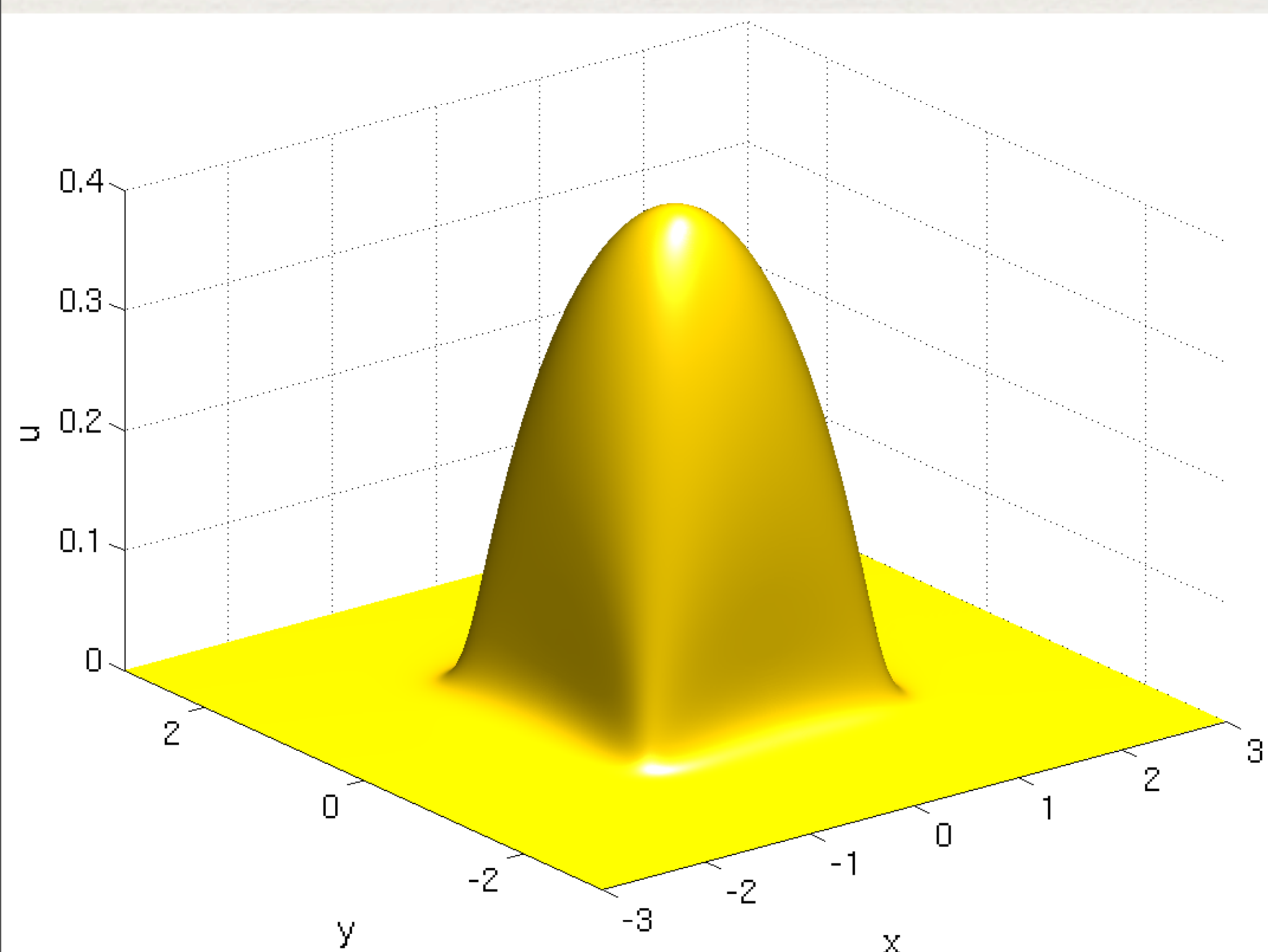
- Ozawa 1992: exact blowup solution for lump-like initial data

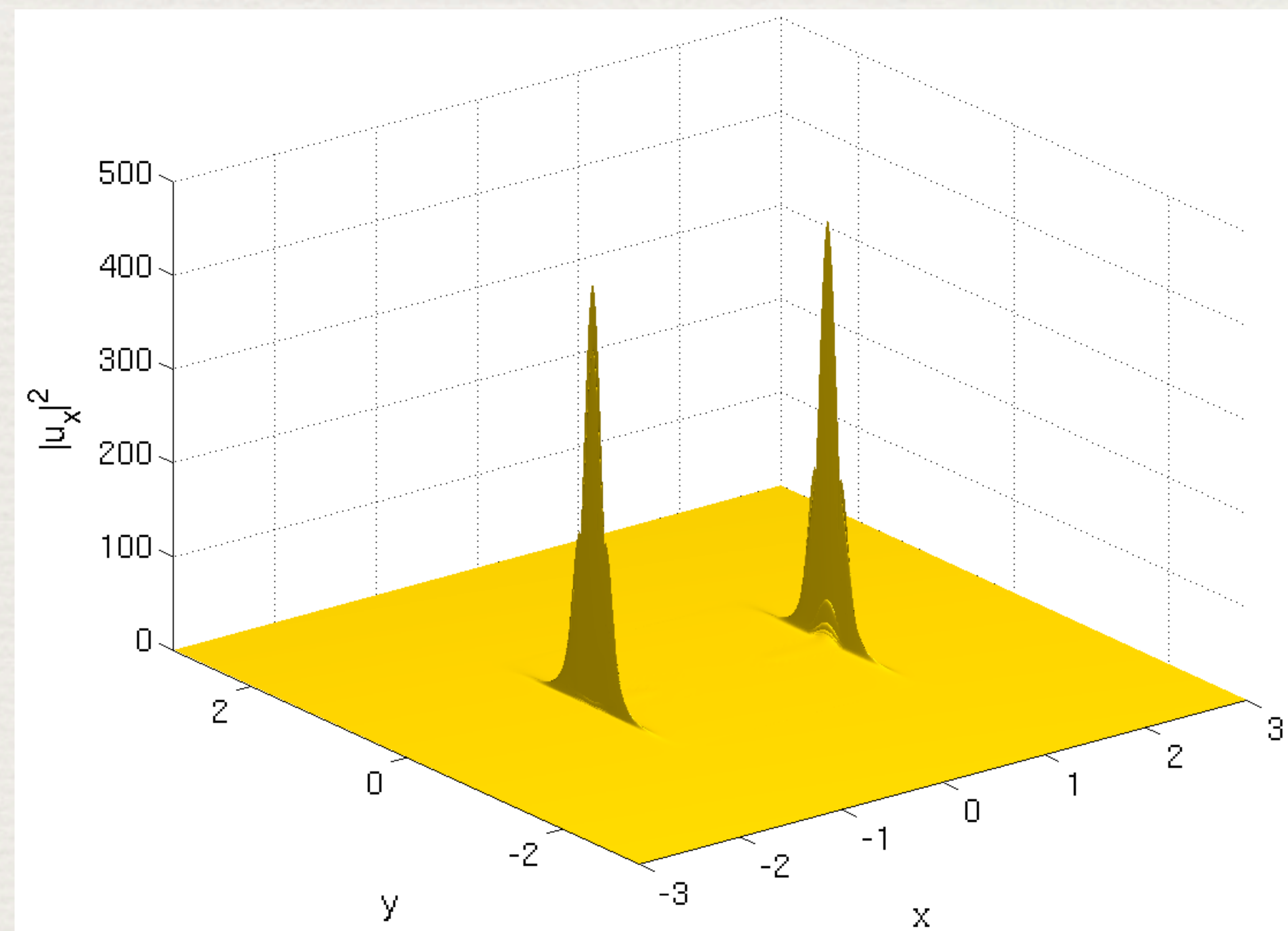
Semiclassical limit

- semiclassical limit ($\Psi = \sqrt{u}e^{iS/\epsilon}$, $\epsilon \to 0$, $\mathcal{D}_\pm = \partial_x^2 \pm \partial_y^2$)

$$\left\{\begin{array}{ccc} S_t + S_x^2 - S_y^2 + 2\rho\mathcal{D}_+^{-1}\mathcal{D}_-(u) & = & \frac{\epsilon^2}{2}\left(\frac{u_{xx}}{u} - \frac{u_x^2}{u^2} - \frac{u_{yy}}{u} + \frac{u_y^2}{u}\right) \\ u_t + 2\left(S_x u\right)_x - 2\left(S_y u\right)_y & = & 0 \end{array}\right. ,$$

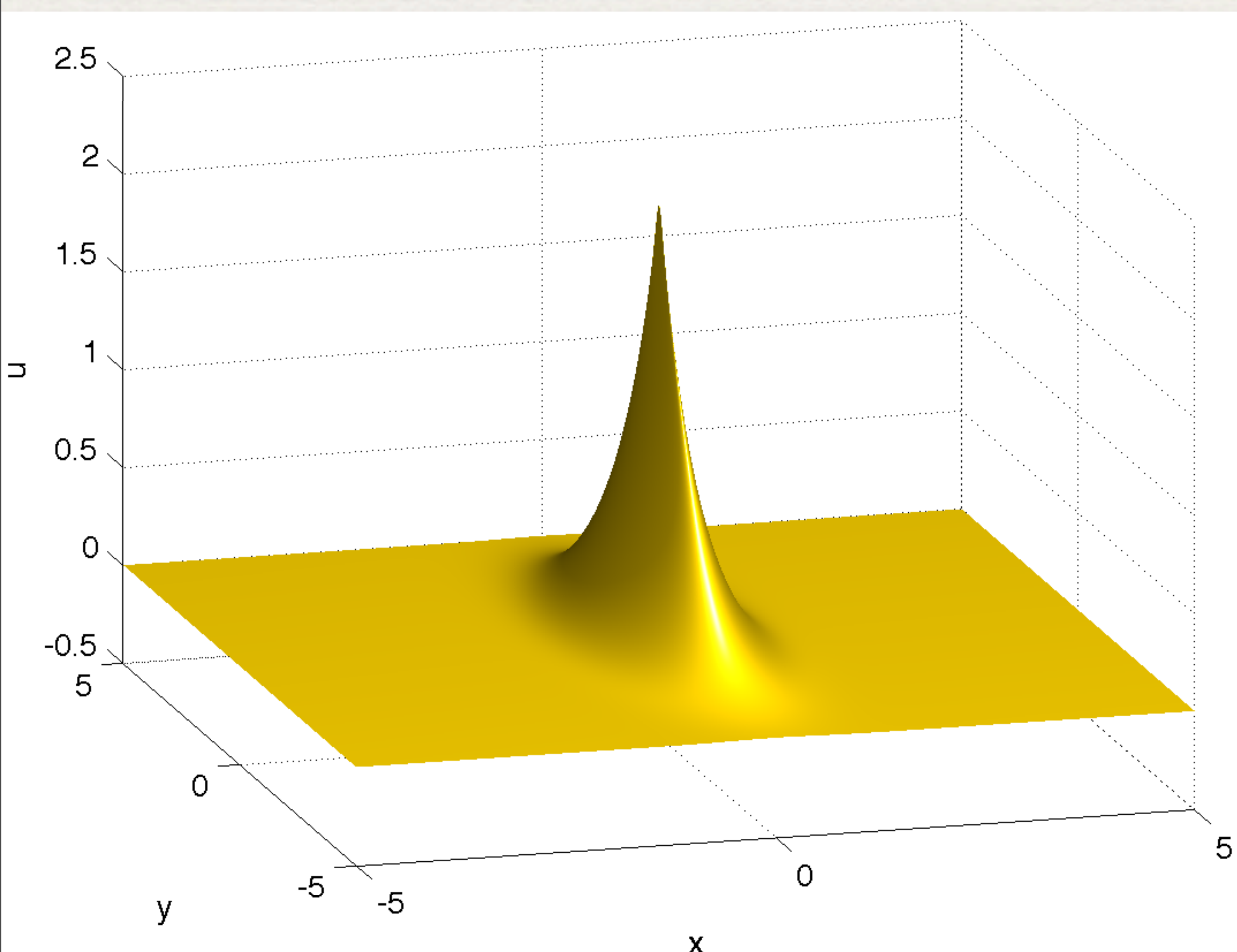
- defocusing case, $u_0 = \exp(-2(x^2+y^2))$, $S_0 = 0$

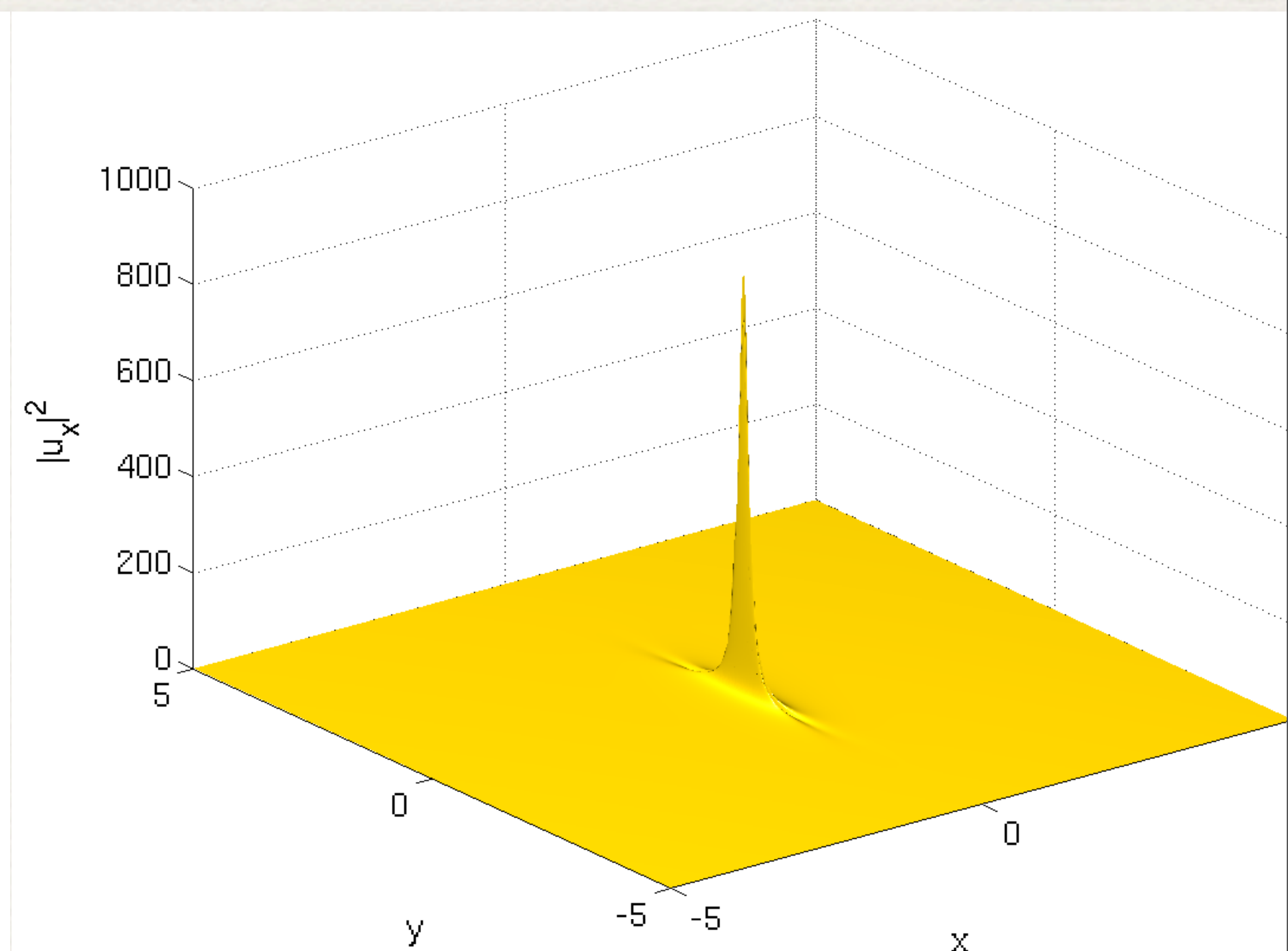




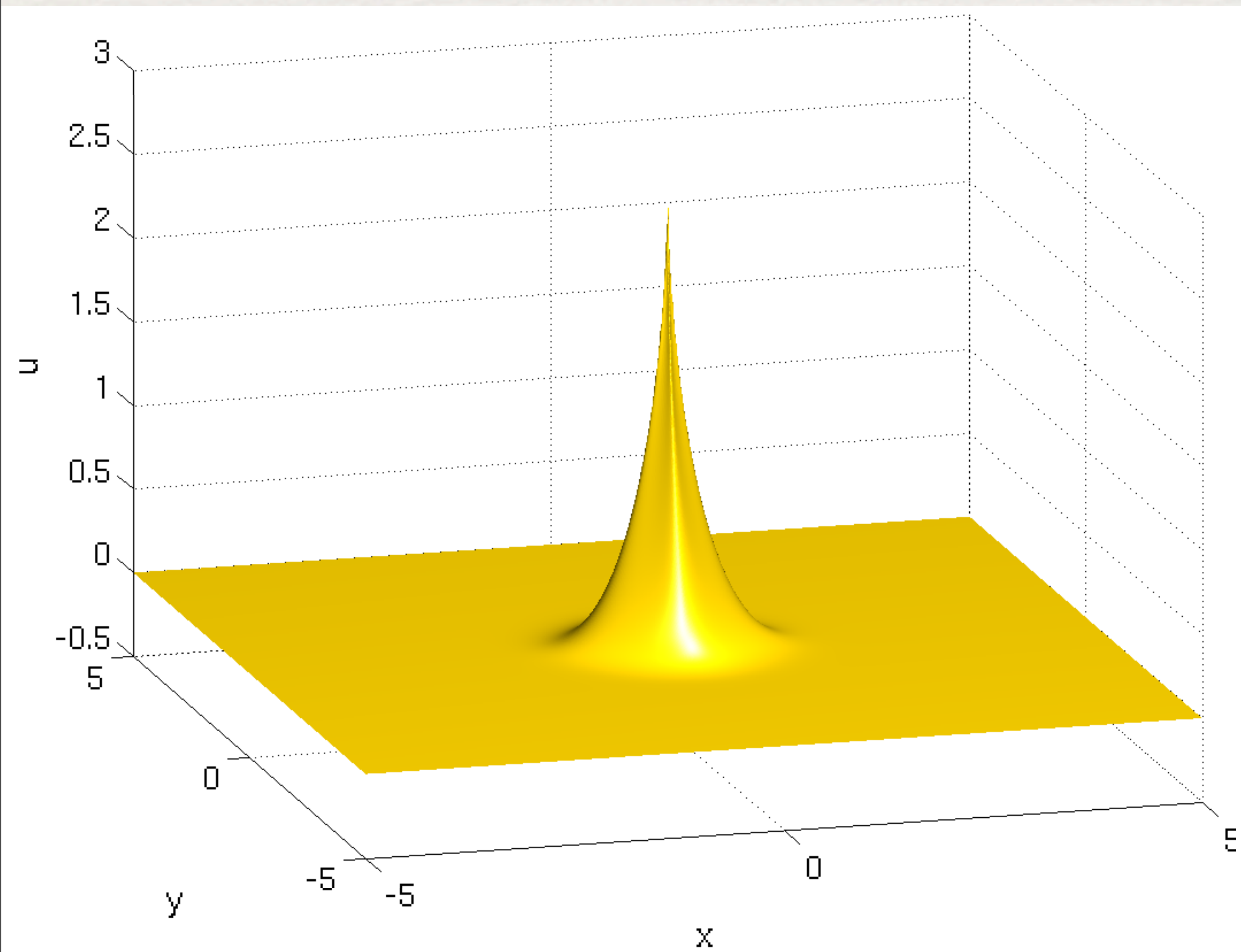
Focusing semiclassical DS II system

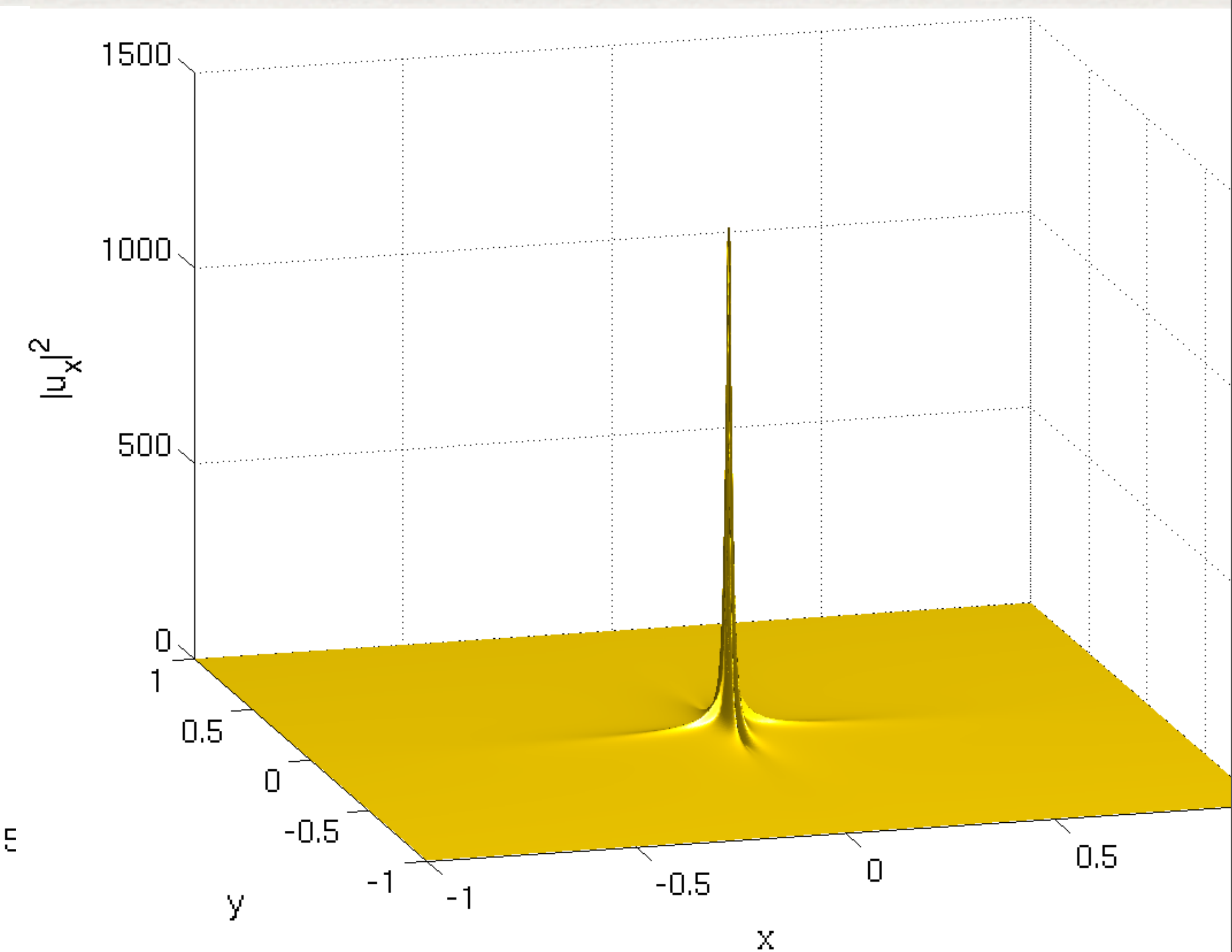
- $u_0 = \exp(-2(x^2 + 0.1y^2)),\ S_0 = 0$





Symmetric initial data





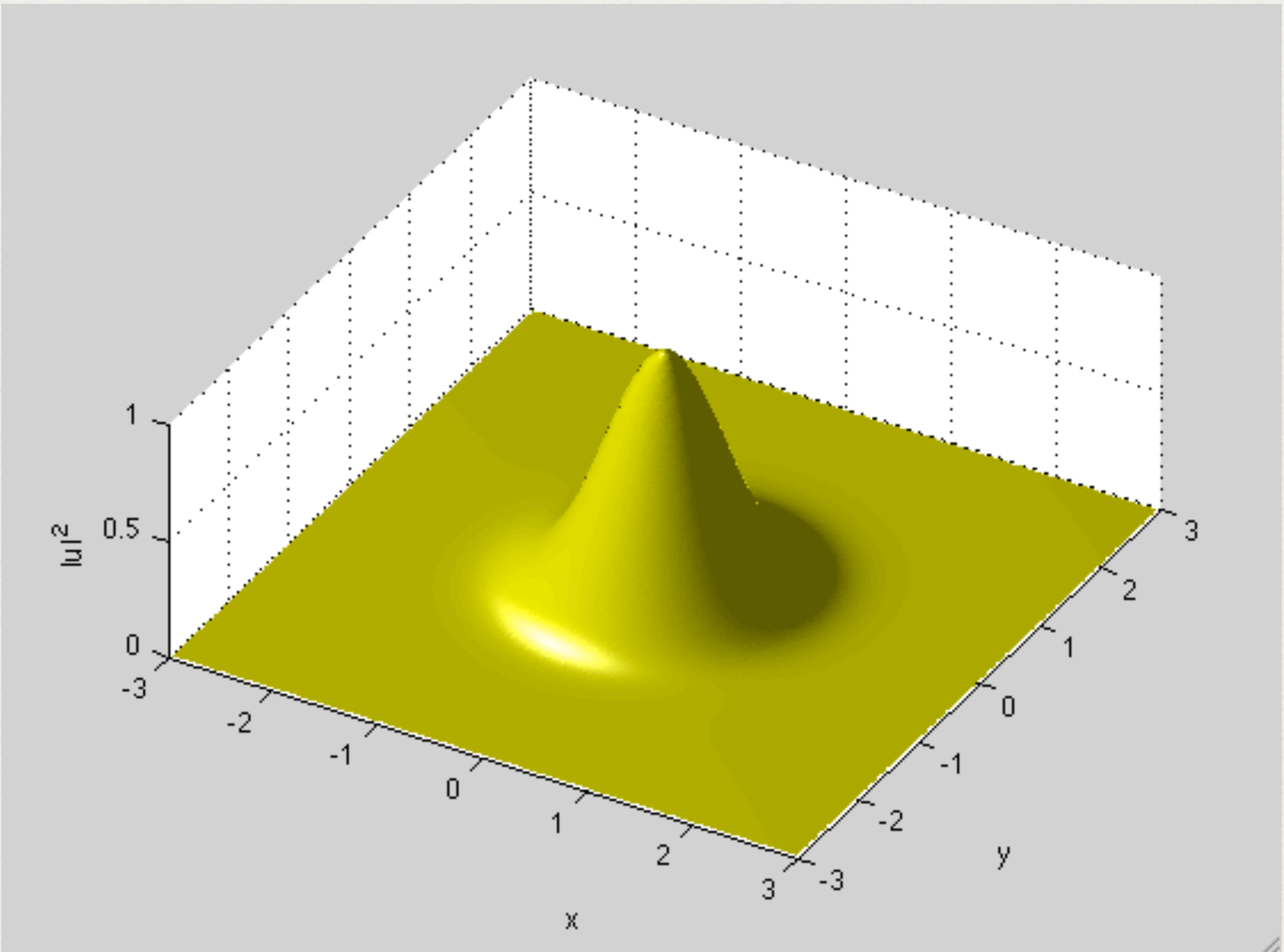
Defocusing DS II

$u_0 = \exp(-x^2 - y^2)$

$\epsilon = 0.1$

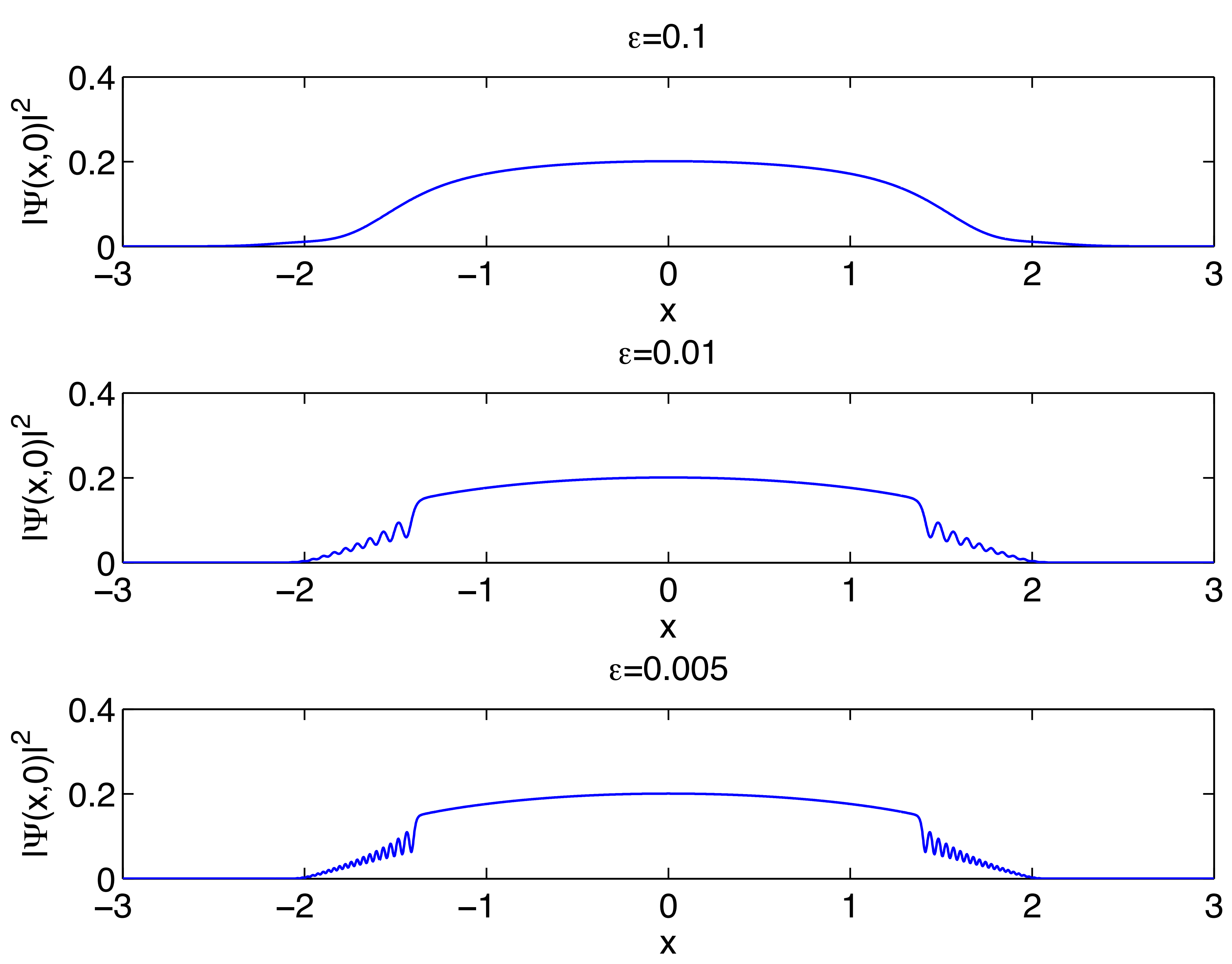
Defocusing DS II

$u_0 = \exp(-x^2 - y^2)$



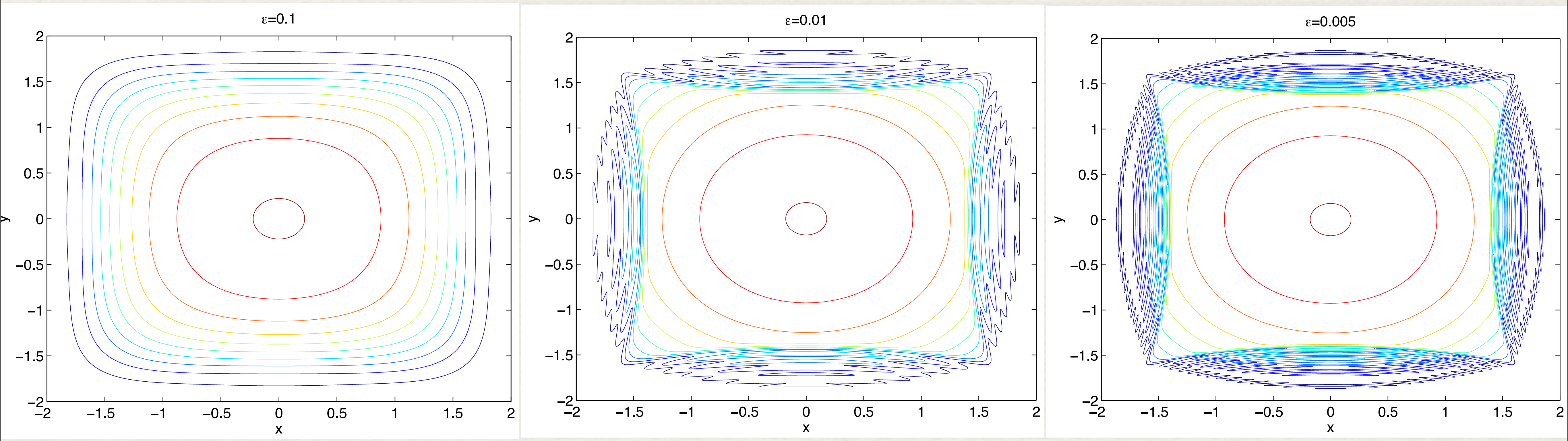
$\epsilon = 0.1$

Defocusing DS II



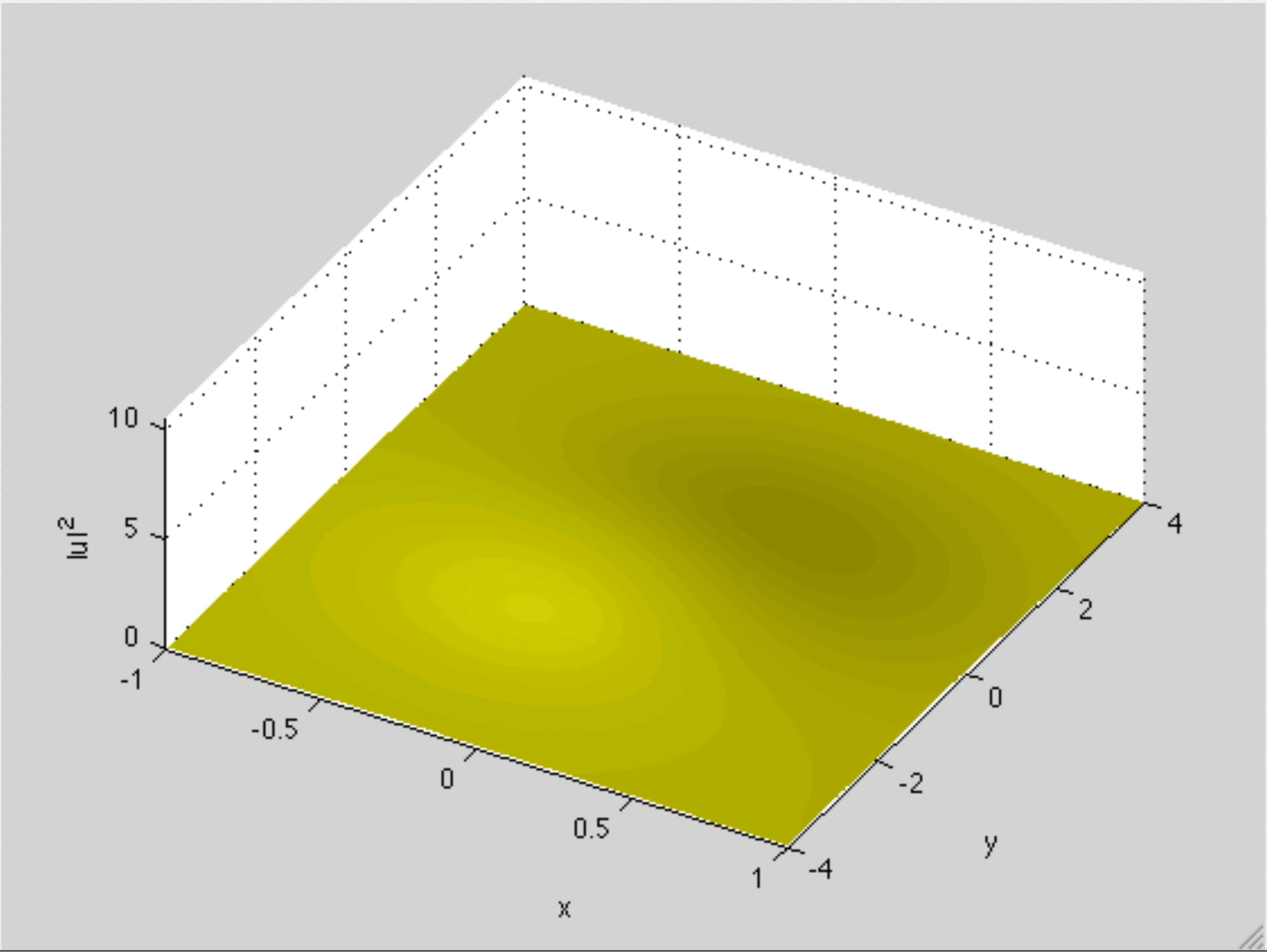
Defocusing DS II

- $t = t_c$: scaling of the difference between semiclassical and DS II solution proportional to $\epsilon^{2/7}$
- $t \gg t_c$: dispersive shock



Focusing DS

$u_0 = \exp(-x^2 - 0.1y^2)$

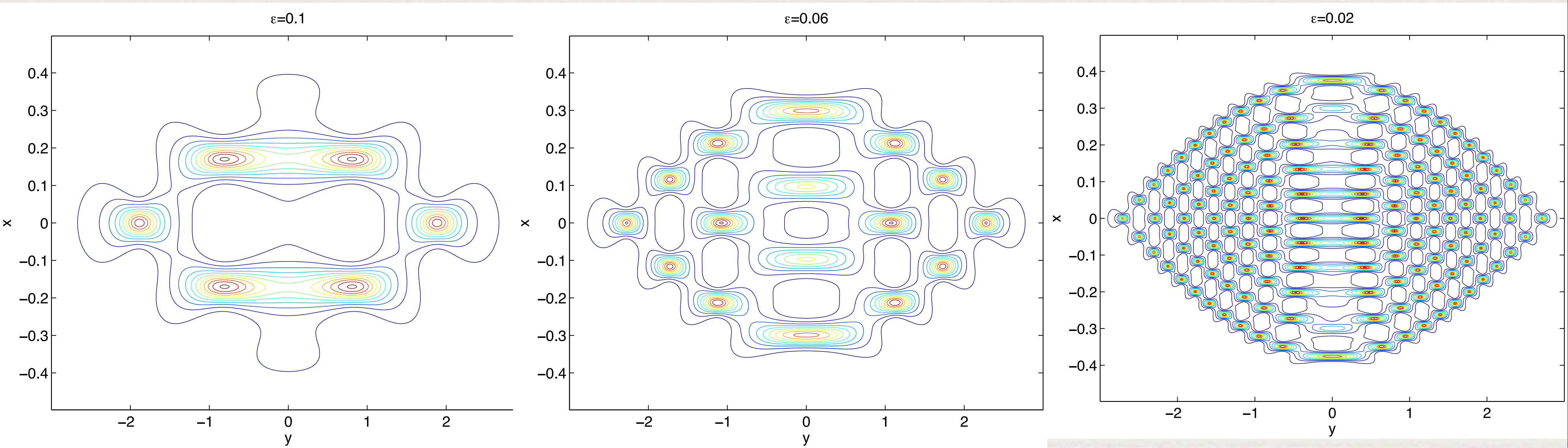


$\epsilon = 0.1$

Focusing DS II

- $t = t_c$: scaling of the difference between semiclassical and DS II solution proportional to $\epsilon^{2/5}$

- $t \gg t_c$: dispersive shock for non-symmetric initial data



Blow-up

- finite time blow-up for symmetric initial data
- not as in Ozawa ($||\Psi||_\infty \propto 1/(t^* - t)$, but as in the stable blow-up for NLS ($||\Psi||_\infty \propto 1/\sqrt{t^* - t}$)
- $t^* - t^c = 0(\epsilon)$